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NEW

UNIVERSITY ALGEBRA:

A

THEORETICAL AND PRACTICAL TREATISE,

DESIGNED FOR USE IN

COLLEGES AND HIGH SCHOOLS.

BY

HORATIO N. ROBINSON, LL. D.,

AUTHOR OF A FULL COURSE OF MATHEMATICS.

NEWLY ELECTROTYPED.

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PREFACE.

IN the preparation of the NEW UNIVERSITY ALGEBRA, care has been taken to preserve every feature of the original work, on which rested, in any degree, its claims to superiority. The aim has been to make that which was *good*, decidedly *better*. Hence the changes that have been made, consist, for the most part, in more apt arrangement, in large additions of original matter, and in presenting the whole in more attractive form.

The treatise, as now submitted to the public, is, indeed, far more complete than the former, not only in the range of topics, but also in general discussions and practical applications. In many parts the methods of investigation are essentially different—the object being, in some instances, to secure simplicity in logical arrangement, and in others, to establish principles and rules by more general and rigorous demonstrations.

The articles on Inequalities, Differential Method of Series, and Interpolation, which, in the old treatise, appear as an appendix, have been elaborated, and made to take their appropriate place in the body of the work.

The section on Radical Quantities is quite full, embracing the more important properties of Imaginary Quantities and Quadratic Surds, besides a complete logical development of the Theory of Exponents.

As, in the author's "New Elementary Algebra," the Binomial Theorem has been fully investigated with reference to integral exponents, it has been deemed unnecessary to repeat here the particular demonstration. Accordingly, the whole subject is deferred till the section on Series is reached, where a general demonstration of this theorem is given in a concise way, and a full variety of applications added. The whole subject, as presented in this connection, with the accompanying illustrations, cannot fail to interest the lovers of Algebra.

The General Theory of Equations is treated in two sections, the one embracing the general properties of equations, and the other the solution

of numerical equations of all degrees. The whole subject is here presented, however, in a condensed form, the student being conducted, in a manner direct as possible, from the theoretical to the practical. The section on the Properties of Equations, it is proper to say, owes its improved character to the able hand of Prof. I. F. QUINBY, of the University of Rochester, whose services in perfecting other books of this Series deserves especial mention.

The effort which has been made in this treatise, to combine the *best practical* with the *highest theoretical* character, is specially commended to the notice of the true educator. Great care has been taken everywhere to set forth in distinct form the *principles* of the science, their exact logical relations being noted by proper references; while due prominence has been given to those numerous precepts and expedients which are so necessary to the constitution of an expert Algebraist.

The design throughout has been, not to *conceal*, but fully to *reveal* the difficulties of the science, and to encourage the learner, not to *avoid*, but to grapple with, and overcome them; since, to the student of Mathematics, labor rightly directed, is *discipline*—and discipline, after all, is the true end of education.

It is but just to state, that J. C. PORTER, A. M., has had the constant care and supervision of the present work, having also rendered important assistance in the preparation of some other works of the Series—a fact which, considering his long and distinguished success as a teacher of Mathematics, and his acknowledged ability as a mathematical scholar, ought to afford a sufficient guarantee for the utmost accuracy and class-room fitness on every page.

Thus distinguished for fullness of matter; for scientific arrangement; for ample discussion and rigid demonstration; for clear statement and close definition; for rules brief and of easy application; for examples numerous, apt, and strictly practical; for the nicest adaptation to the purposes of teaching; for the finest mechanical execution; for whatever, in short, care, skill, science, and taste can accomplish;—the NEW UNIVERSITY ALGEBRA is submitted to the public.

SEPTEMBER, 1875.

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SECTION I.

DEFINITIONS AND NOTATION.

1. *Quantity* is anything that can be increased, diminished, or measured; as distance, space, weight, motion, time.

A quantity is measured by finding how many times it contains a certain other quantity of the same kind, regarded as a standard. The conventional standard thus used is called the *unit of measure.*

2. *Mathematics* is the science which treats of the properties and relations of quantities. It employs a peculiar language, consisting of symbols, to express the values of quantities, and the operations to which these values are subjected. The symbols are of three kinds, as follows:

1st. *Symbols of Quantity,* consisting of figures or numerals used in arithmetical computations, letters and other characters used in general analysis, and graphic representations or drawings used in geometrical investigations.

2d. *Symbols of Operation,* consisting of the signs or characters employed to indicate those mathematical processes by which quantities are made to undergo changes of value, such as addition, subtraction, multiplication, and division.

3d. *Symbols of Relation,* consisting of the signs used in comparing quantities with respect to their relative magnitudes, and certain abbreviations employed in the process of reasoning.

3. *Algebra* is that branch of mathematics in which quantities are represented by letters, and the operations and relations are indicated by signs. The object of algebraic notation is to abridge and generalize the analysis of mathematical problems. Algebra is therefore a species of *universal arithmetic.*

SYMBOLS OF QUANTITY.

4. An ***Algebraic Quantity*** is a quantity expressed in algebraic language. There are two kinds of algebraic quantities —*known* and *unknown.*

5. ***Known Quantities*** are those whose values are given; when these are not expressed by figures they are represented by the leading letters of the alphabet, as a, b, c, d.

6. ***Unknown Quantities*** are those whose values are to be determined; they are represented by the final letters of the alphabet, as u, x, y, z.

7. The small italic letters just given are the more common symbols of quantity. In addition to these, capital letters are sometimes employed, as A, B, C, D, X, Y, Z.

Quantities which have like relations to a series of quantities in any investigation, are sometimes represented by a single letter repeated with different accents, as a, a', a'', a''', a'''', read, a, a prime, a second, a third, etc.; or by a letter repeated with different subscript figures, as a, a_1, a_2, a_3, a_4, etc., read, a, a sub one, a sub two, a sub three, etc.

In certain investigations it is convenient to represent quantities by the initial letters of their names. Thus, S or s may represent *sum; D* or *d, difference* or *diameter; R* or *r, ratio, remainder,* or *radius.* In some cases the capital and the small letter may be used together to distinguish between two quantities of the same kind. Thus, in a problem relating to two circles, r may represent the radius of the smaller, and R the radius of the larger circle.

SYMBOLS OF OPERATION.

8. The ***Sign of Addition*** is the perpendicular cross, $+$, called *plus.* It indicates that the quantity written after it is to be added to the other quantity or quantities in the expression. Thus, in $a + b$, the sign indicates that the quantity b is to be added to the quantity a; and the expression is read, *a plus b.*

9. The ***Sign of Subtraction*** is a short horizontal line, $-$, called *minus.* It indicates that the quantity written after it is to be subtracted from the other quantity or quantities in the

expression. Thus, in $a - b$, or $-b + a$, the minus sign indicates that the quantity b is to be subtracted from the quantity a; and the expression is read, *a minus b*, or *minus b plus a*.

The sign $\sim$ may be written between two quantities to indicate that their arithmetical difference is to be taken, when it is not known which is the greater.

10. The ***Double Sign***, $\pm$, is written before a quantity to indicate that it is to be added or subtracted; it serves to unite in a single expression two combinations of the same quantities. Thus, $a \pm b$ is equivalent to $a+b$ or $a-b$, and is read, *a plus or minus b*.

11. The ***Sign of Multiplication*** is the oblique cross, $\times$. It indicates that the quantity before it is to be multiplied by the quantity after it. Thus, in $a \times b$, the sign indicates that a is to be multiplied by b. Instead of the sign, $\times$, a point is sometimes used to denote multiplication; as $3 \cdot x \cdot y$, which signifies the same as $3 \times x \times y$.

The multiplication of quantities which are represented by letters, is generally indicated by writing the factors one after another without any intervening sign. Thus, $3abc$ signifies the same as $3 \times a \times b \times c$, or $3 \cdot a \cdot b \cdot c$. It is evident that this notation cannot be employed when the several factors are represented by figures. We cannot represent 3 times 4 by simply writing the factors together, thus, 3 4; for the product thus indicated could not be distinguished from the number 34.

NOTES.—1. The result of any multiplication is called a *product*, and the quantities multiplied are called *factors*.

2. When the quantities to be multiplied are represented by letters, they are called *literal factors*; when they are represented by figures, they are called *numerical factors*.

12. The ***Sign of Division*** is a short horizontal line with a point above and one below, $\div$. It indicates that the quantity before it is to be divided by the quantity after it. Thus, in $a \div b$, the sign indicates that a is to be divided by b.

Division is also expressed by writing the dividend above, and the divisor below, a short horizontal line; as $\frac{a}{b}$.

13. The ***Sign of Involution*** is a number written above and to the right of a quantity, to indicate how many times the

quantity is to be taken as a factor. Thus, in a^5, the number 5 indicates that a is to be taken 5 times as a factor; and the expression is equivalent to $aaaaa$.

A factor repeated to form a product is called a *root;* the product itself is called a *power;* and the figure which indicates how many times the root or factor is taken, is called the *exponent* of the power. Thus, in the indicated product a^5, a is the root, a^5 is the power, called the 5th power of a, and 5 is the exponent of this power. When no exponent is written over a quantity, the exponent 1 is understood.

NOTE.—For the sake of brevity, the exponent of the power may be called the exponent of the letter or quantity over which it is placed. Thus, in a^5, 5 may be called the exponent of a.

14. The ***Sign of Evolution***, or ***Radical Sign***, is the character $\sqrt{\ }$. It indicates that some root of the quantity after it is to be extracted. The name or *index* of the required root is the number written above the radical sign. Thus, $\sqrt[3]{a}$ denotes the cube root of a; $\sqrt[4]{a}$ denotes the 4th root of a; and so on. When no index is written over the sign, the index 2 is understood; thus, $\sqrt{a}$ denotes the square root of a.

Fractional Exponents are also used as the sign of evolution, the denominator being the index of the required root. Thus, in $a^{\frac{1}{3}}$, the denominator, 3, indicates that the cube root of a is required, and the expression is equivalent to $\sqrt[3]{a}$.

Fractional exponents are used to denote both involution and evolution in the same expression, the numerator indicating the *power* to which the quantity is to be raised, and the denominator the required *root* of this power. Thus, the expression $a^{\frac{3}{4}}$ signifies the 4th root of the third power of a, and is equivalent to $\sqrt[4]{a^3}$.

SYMBOLS OF RELATION.

15. The ***Sign of Equality*** is two short horizontal lines, $=$. It indicates that the two quantities between which it is placed are equal. Thus, in $a = b + c$, the sign, $=$, indicates that a is equal to b plus c. An expression of equality between two quantities is called an *equation.*

16. The ***Sign of Inequality*** is the angle, $>$. It indicates that the quantities between which it is written are unequal, the opening being turned toward the greater. When the opening is toward the left, it is read *greater than;* when the point or vertex is toward the left, it is read *less than.* Thus, $a>b$ signifies that a is greater than b; $x+y<z$ signifies that x plus y is less than z.

17. The ***Signs of Aggregation*** are the parenthesis, ( ), brackets, [ ], brace, { }, vinculum, ———, and bar, |. They indicate that the quantities included within, or connected by them, are to be taken collectively and subjected to the same operation. Thus, $(a+b-c)x$, $[a+b-c]x$, $\{a+b-c\}x$, $\overline{a+b-c}\times x$,

and $\begin{array}{r|l} +a & x \\ +b & \\ -c & \end{array}$ are expressions signifying that the whole quantity, $a+b-c$, is to be multiplied by x. Two or more of these signs may be used in the same expression, in which case the brackets usually include the parenthesis or vinculum, and the brace includes the bracket; thus,

$$\{m-a[c-b(m+d)]+z\}.$$

18. The ***Sign of Continuation*** is a succession of points, indicating that a series of quantities may be continued indefinitely according to the same law. Thus, in the expression, $a+a^2+a^3+a^4+\ldots\ldots$, the points indicate that the series has an infinite number of terms, all formed according to the same law.

19. The ***Sign of Ratio*** is two points like the colon, :, placed between the quantities compared. Thus, the expression, $a:b$, signifies the ratio of a to b.

20. The ***Sign of Proportion*** is a combination of the sign of ratio and the sign of equality, : = : ; or a combination of points only, : :: :. Thus, $a:b=c:d$, signifies that the ratio of a to b is equal to the ratio of c to d; and the expression $a:b::c:d$ signifies the same, and may be read, *a is to b as c is to d.*

21. The ***Sign of Variation*** is the character $\propto$. It signifies that the two quantities between which it is placed, whether equal or unequal, increase or diminish together, so as to preserve constantly the same ratio.

NOTE.—The signs of ratio, proportion, and variation, will be more fully explained hereafter.

COMPOSITION OF ALGEBRAIC QUANTITIES.

22. An algebraic quantity may consist of a single letter or element, or a combination of symbols as factors, or several combinations or parts. The parts are called *terms;* hence,

23. The ***Terms*** of an algebraic quantity are the parts or divisions made by the signs $+$ and $-$. Thus, in the quantity $5a + 2b^2 - cx$, there are three terms, of which $5a$ is the first, $2b^2$ the second, and cx the third.

24. When a quantity consists of a single term, it is said to be *simple;* when it is composed of two or more terms, it is said to be *compound.*

25. ***Positive Terms*** are those which have the plus sign; as $+x$, or $+2c^2d$. The first term of an algebraic quantity, if written without any sign, is positive, the plus sign being understood.

26. ***Negative Terms*** are those which have the minus sign; as $-3a$, or $-2mx^2$. The sign of a negative quantity is never omitted.

27. A ***Coefficient*** is a number or quantity prefixed to another quantity, to denote how many times the latter is taken. Thus, in $3x$, the number 3 is the coefficient of x, and indicates that x is taken 3 times; hence, the expression $3x$ is equivalent to $x+x+x$. In $4ax$, 4 may be regarded as the coefficient of ax, or $4a$ as the coefficient of x. In $5(a+x)$, 5 is the coefficient of $a+x$. When no coefficient is written, the unit 1 is understood.

28. It should be observed that in a term having the plus sign, the coefficient shows how many times the quantity is to be added; and in a term having the minus sign, the coefficient shows how many times the quantity is to be subtracted. Thus,

$$+3a = +a + a + a$$
$$-3a = -a - a - a.$$

29. ***Similar Terms*** are terms containing the same letters, affected with the same exponents; the signs and coefficients may differ, and the terms still be similar. Thus, $3x^2$ and $7x^2$ are similar terms; also, $2md^2$ and $-5md^2$ are similar terms.

30. ***Dissimilar Terms*** are those which have different letters or exponents. Thus, axy and ayz are dissimilar; also $3x^2y$ and $3x^3y^2$.

31. A ***Monomial*** is an algebraic quantity consisting of only one term; as $3x$, or $-7xy$.

32. A ***Polynomial*** is an algebraic quantity consisting of more than one term; as $x+y$, or $4a^2-3x+m$.

33. A ***Binomial*** is a polynomial of two terms; as $a+b$, or $3x-z$.

34. A ***Residual*** is a binomial, the two terms of which are connected by the minus sign; as $a-b$, or $4x-3y$.

35. A ***Trinomial*** is a polynomial of three terms; as $x+y+z$, or $7a-3b^2+d$.

36. The ***Degree*** of a term is the number of its literal factors. Since the exponents show how many times the different letters are taken as factors, the degree of a term is always found by adding the exponents of all the letters. Thus, x and $5y$ are terms of the first degree; a^2 and $4ab$ are terms of the second degree; x^3, $3x^2y$, $3xy^2$, and $4xyz$ are terms of the third degree.

37. A ***Homogeneous Quantity*** is one whose terms are all of the same degree; as $x^3-5x^2y+3xyz$.

38. A ***Function*** of a quantity is any expression containing that quantity. Thus ax^4 is a function of x; $3y^2+2y-4$ is a function of y.

AXIOMS.

39. An ***Axiom*** is a self-evident truth. The following axioms underlie the principles of all algebraic operations:

1. If the same quantity or equal quantities be added to equal quantities, the sums will be equal.
2. If the same quantity or equal quantities be subtracted from equal quantities, the remainders will be equal.
3. If equal quantities be multiplied by the same, or equal quantities, the products will be equal.
4. If equal quantities be divided by the same, or equal quantities, the quotients will be equal.
5. If a quantity be both increased and diminished by another, its value will not be changed.
6. If a quantity be both multiplied and divided by another, its value will not be changed.

7. Quantities which are respectively equal to the same quantity, are equal to each other.

8. Like powers of equal quantities are equal.

9. Like roots of equal quantities are equal.

10. The whole of a quantity is greater than any of its parts.

11. The whole of a quantity is equal to the sum of all its parts.

EXERCISES IN ALGEBRAIC NOTATION.

40. In the examples which follow, it is required of the pupil simply to express given relations in algebraic language.

1. Give the algebraic expression for the square of a increased by 4 times b. *Ans.* $a^2 + 4b$.

2. Give the algebraic expression for 7 times the product of x and y, diminished by 5 times the cube of z.

3. Indicate the quotient of 12 times the square of a minus 5 times the cube of b, divided by the sum of a and c.

4. If d represent a person's daily wages, what will represent his wages for 6 days? *Ans.* $6d$.

5. An army drawn up in rectangular form, has b men in rank, and a men in file; of how many men is the army composed?

6. If a man labor m days in a week at c dollars per day, what will his earnings amount to in 7 weeks?

7. The length of a prism is a, the breadth c, and the altitude $a - c$; required the solid contents. *Ans.* $ac(a - c)$.

8. A has $4m$ dollars, B has m times as many dollars as A, and C has 3 times as many dollars as B wanting d dollars; how many dollars has C?

9. A dealer sells b sheep and c calves, at an average price of m dollars per head; how much does he receive for all?

10. A man has 3 square lots measuring m rods on a side; how many acres in the three lots? *Ans.* $\frac{3m^2}{160}$.

11. From a rectangular piece of land whose length was a rods and whose width was b rods, there were sold c acres; how many acres remained unsold?

12. A ship laden with a barrels of flour, valued at m dollars

per barrel, met with a disaster by which b barrels were lost, and the remainder damaged to the amount of d dollars per barrel; what was the worth of the remainder? *Ans.* $(a-b)(m-d)$.

13. A man having c acres of land worth b dollars per acre, divided its value equally among m sons and one daughter; how many dollars did each receive?

14. A company of n persons began business with a joint capital of c dollars. The first year they gained b dollars, the second year they lost d dollars, the third year they doubled the capital with which they began that year, and then dissolved partnership, sharing equally their accumulated capital; what was each man's share?

COMPUTATION OF NUMERICAL VALUES.

41. The ***Numerical Value*** of an algebraic quantity is the number obtained by assigning numerical values to all the letters, and performing the operations indicated.

1. What is the numerical value of $(a^2 - bc)a$, when $a = 30$, $b = 25$, and $c = 28$?

OPERATION.

$$(a^2 - bc)a = (30 \times 30 - 25 \times 28) \times 30 = 200 \times 30 = 6000, \text{ } Ans.$$

Find the numerical values of the following expressions, in which

$$a = 12;\ b = 10;\ c = 8;\ m = 6;\ n = 5;\ d = 2.$$

2. $a^2 - bc$. *Ans.* 64.
3. $(a + bd)m$. *Ans.* 192.
4. $am + c^2 - md^4$. *Ans.* 40.
5. $(a + b)m - (c + d)n$. *Ans.* 82.
6. $[4a^2 - (3b^2 - 2c)]d$. *Ans.* 584.
7. $(a^2 - b)(b^2 - a)$. *Ans.* 11792.
8. $\dfrac{4a - c}{n}$. *Ans.* 8.
9. $\dfrac{2b^2 + c^2}{md^2} \times (a - c)$. *Ans.* 44.
10. $\dfrac{3am - (b^2 + 2c)}{2b + n} + \dfrac{a^2 - d^2}{3n - c}$. *Ans.* 24.

Find the numerical values of the following expressions, in which
$a = 8;\ b = 6;\ c = 4;\ d = 2;\ m = 3;\ n = 1.$

11. $\dfrac{a^2 + b^2 + c^2 + d^2}{a + b + c + d}.$ *Ans.* 6.

12. $(6a^2n - 4m^3d)(m^2 - 7n).$ *Ans.* 336.

13. $\dfrac{(5an + 26)\,d^2}{a + b + c + d + m + n}.$ *Ans.* 11.

14. $\left(\dfrac{5a}{c} - \dfrac{3c}{d^2}\right)c.$ *Ans.* 28.

15. $\left\{2c\,(a^2d - m^3) - 2\,(5b^2 + 4m) + c\right\}d^3.$ *Ans.* 3424.

16. $\dfrac{(a + b - c)\,(a - b + c)}{m^2 - c\,(n + 1) + 1}.$ *Ans.* 30.

17. $\dfrac{1}{m}\left(\dfrac{am^2}{c} + \dfrac{d^4 - 1}{m^2 - b}\right).$ *Ans.* $7\frac{2}{3}$.

18. $\dfrac{1}{c}\left\{\left[\overline{a + 2c} \times m - d^2\right]m - 2\,(ab + m^2)\right\}.$ *Ans.* $4\frac{1}{2}$.

19. $\left(\dfrac{a}{b} + \dfrac{m}{c} + \dfrac{d}{m}\right)b.$ *Ans.* $16\frac{1}{2}$.

20. $\dfrac{abcd + a + b + c + d}{m + n}.$ *Ans.* 101.

21. $\left(\dfrac{a}{a + b} + \dfrac{b}{a - b}\right) \div \left(\dfrac{a}{a - b} - \dfrac{b}{a + b}\right).$ *Ans.* 1.

22. $\dfrac{6a^2 - 22a + 18}{a^3 - 6a^2 + 11a - 6}.$ *Ans.* $8\frac{8}{105}$.

23. $\dfrac{1}{a - 1} + \dfrac{2}{a - 2} + \dfrac{3}{a - 3}.$ *Ans.* $1\frac{8}{105}$.

SIGNIFICATION OF THE PLUS AND MINUS SIGNS.

42. The signs, $+$ and $-$, have been defined as *symbols of operation,* the former indicating addition, and the latter subtraction. Now when we meet with detached or single terms affected with the plus or minus signs, as for instance in the examples of addition, subtraction, multiplication, or division,

we are to consider the positive terms as quantities to be *added*, and the negative terms as quantities to be *subtracted;* and this is the only signification that need be attached to these signs, at present.

It is obvious that positive and negative terms, as just explained, are, in an important sense, opposites; and it will be shown in a future section (**182**) that quantities sustaining to each other various other relations of opposition or contrariety are distinguished by the plus and minus signs.

43. In order to establish general rules for algebraic operations, it will be necessary in this place to recognize the following principles, consequent upon the peculiar manner of considering quantities in Algebra:

1. A quantity may be considered as having two kinds of value —an absolute or numerical value determined simply by the number of units it contains, and an algebraic value depending on the sign.

2. Two quantities having the same absolute value, but affected with unlike signs, are not algebraically equal. Thus, $5a$ is not equal to $-5a$, for the former expression signifies that a is to be added five times, and the latter signifies that a is to be subtracted five times.

3. Two quantities having the same absolute value, but affected with unlike signs, are together equal to zero. Thus, if a denote the absolute value of any quantity, then

$$+a-a=0$$
$$+2a-2a=0$$
$$+3a-3a=0, \text{ etc.}$$

ADDITION.

44. ***Addition*** is the process of uniting two or more quantities into one equivalent expression called their *sum.*

45. The term, addition, has a more general meaning in Algebra than in Arithmetic, because the quantities to be added may be either positive or negative.

46. The ***Arithmetical Sum*** of two or more quantities is the sum of their absolute values, and has reference simply to the number of units in the quantities added.

47. The ***Algebraic Sum*** of two or more quantities is a quantity which, taken with reference to its sign, is equivalent to the given quantities, each taken with reference to its particular sign.

48. To deduce a rule for addition, which will conform to the nature of positive and negative quantities, let us consider the following examples:

1. Add $4a$, $3a$, and $5a$.

Since in these quantities a is taken positively, 4, 3, and 5 times, or 12 times, the algebraic sum required must be $+ 12a$; or simply, $12a$. That is,

$$4a + 3a + 5a = 12a.$$

2. Add $-4a$, $-3a$, and $-5a$.

Since in these quantities a is taken negatively, 4, 3, and 5 times, or 12 times, the algebraic sum required must be $-12a$. That is,

$$-4a - 3a - 5a = -12a.$$

Hence,

The algebraic sum of two or more similar terms having like signs, is the sum of their absolute values taken with their common sign.

3. Add $7a$ and $-3a$.

From Ax. 11 we have

$$7a = 4a + 3a.$$

The sum of $7a$ and $-3a$ is therefore the same as the sum of $4a$, $3a$, and $-3a$. But since $3a$ and $-3a$ taken together are equal to nothing (**43**, 3), the required sum must be the remaining term, $4a$. That is,

$$7a + (-3a) = 4a + 3a - 3a = 4a, \quad \textit{Ans.}$$

4. Add $-7a$ and $3a$.

From Ax. 11,

$$-7a = -4a - 3a.$$

The sum of $-7a$ and $3a$ is therefore the same as the sum of $-4a$, $-3a$, and $3a$. But, since $-3a$ and $+3a$ taken together are equal to nothing (**43**, 3), the required sum must be the remaining term, $-4a$. That is,

$$-7a + 3a = -4a - 3a + 3a = -4a, \textit{ Ans.}$$

Hence,

The algebraic sum of two similar terms having unlike signs, is the difference of their absolute values taken with the sign of the greater term.

It may further be observed,

1st. That three or more similar terms, having different signs, may be added, by first finding the sum of the positive terms, and the sum of the negative terms, separately, and then adding these results. Thus,

$$3a - 5a - 4a + 2a + 8a = 13a - 9a = 4a.$$

2d. Dissimilar terms cannot be united into one term by addition, because the quantities have not a common unit. We can therefore only *indicate* the addition of dissimilar terms, by connecting them by their respective signs. Thus, the sum of a, b, and $-c$ is

$$a + b - c.$$

3d. It is immaterial in what *order* the terms of an algebraic quantity are written, the value being the same so long as the signs of the terms remain unchanged. Thus,

$$a + b - c = b + a - c = a - c + b = -c + a + b.$$

For, each of these expressions denotes the sum of the three terms, a, b, and $-c$.

49. From these principles and illustrations we deduce the following

Rule.—To add similar terms:

I. *When the signs are alike, add the coefficients, and prefix the sum, with the given sign, to the common literal part.*

II. *When the signs are unlike, find the sum of the positive and of the negative coefficients separately, and prefix the difference of the two sums, with the sign of the greater, to the common literal part.*

To add polynomials:

I. *Write the quantities to be added, placing the similar terms together in separate columns.*

II. *Add each column, and connect the several results by their respective signs.*

EXAMPLES FOR PRACTICE.

(1.)
$$\begin{array}{r} 3xy \\ xy \\ 4xy \\ 6xy \\ \hline 14xy. \end{array}$$

(2.)
$$\begin{array}{r} -6a^2b \\ -2a^2b \\ -\ a^2b \\ -8a^2b \\ \hline -17a^2b. \end{array}$$

(3.)
$$\begin{array}{r} +\ a^2cx \\ -4a^2cx \\ +6a^2cx \\ -\ a^2cx \\ \hline +2a^2cx. \end{array}$$

(4.)
$$\begin{array}{r} -9x^2yz \\ x^2yz \\ 4x^2yz \\ -3x^2yz \\ \hline -7x^2yz. \end{array}$$

(5.)
$$\begin{array}{r} 4x^2-3xy \\ x^2+2xy \\ 2x^2-\ xy \\ 3x^2+5xy \\ 5x^2-4xy \\ \hline 15x^2-\ xy. \end{array}$$

(6.)
$$\begin{array}{r} -7a^2c+\ m \\ +4a^2c-3m \\ -3a^2c+5m \\ +\ a^2c-2m \\ +9a^2c+4m \\ \hline 4a^2c+5m. \end{array}$$

(7.)
$$\begin{array}{r} 3a-2\sqrt{c} \\ 4a+3\sqrt{c} \\ a-7\sqrt{c} \\ 5a+3\sqrt{c} \\ 2a-\ \sqrt{c} \\ \hline 15a-4\sqrt{c} \end{array}$$

(8.)
$$\begin{array}{r} 4(c-2a)-\ m+\ 4 \\ 3(c-2a)+4m-\ 8 \\ -8(c-2a)-3m+12 \\ 12(c-2a)+\ m-16 \\ \hline 11(c-2a)+\ m-\ 8. \end{array}$$

(9.)
$$\begin{array}{r} 5(a-x^2)+3\sqrt{a-x}+\ 5 \\ 4(a-x^2)-2\sqrt{a-x}+\ 8 \\ 2(a-x^2)-8\sqrt{a-x}-12 \\ -(a-x^2)+2\sqrt{a-x}-\ 1 \\ \hline 10(a-x^2)-5\sqrt{a-x}. \end{array}$$

10. Add $12a^2x$, $5a^2x$, $-4a^2x$, $6a^2x$, and $-10a^2x$. *Ans.* $9a^2x$.

11. Add $4abd^2$, $-2abd^2$, $7abd^2$, abd^2, $-5abd^2$, $-13abd^2$, and $7abd^2$. *Ans.* $-abd^2$.

12. Add $2xy-2a^2$, $3a^2+2xy$, a^2+xy, $4a^2-3xy$, and $2xy-2a^2$. *Ans.* $4a^2+4xy$.

13. Add $8a^2x^2-3xy$, $5ax-5xy$, $9xy-5ax$, $2a^2x^2+xy$, and $5ax-3xy$. *Ans.* $10a^2x^2-xy+5ax$.

14. Add $a^2-2ac+cd+b$, $3a^2-3ac-3cd-2b$, $2a^2+ac-5cd+6b$, and $a^2-4ac+2cd-3b$. *Ans.* $7a^2-8ac-5cd+2b$.

15. Add $2a^2x^2-3mx+4m^2d$, $3m^2d+5a^2x^2-5mx$, $6mx-4m^2d-3a^2x^2$, and $2mx-3a^2x^2-3m^2d$. *Ans.* a^2x^2.

16. Add $2bx-12$, $3x^2-2bx$, $5x^2-3\sqrt{x}$, $3\sqrt{x}+12$, and x^2+3. *Ans.* $9x^2+3$.

17. Add $10b^2-3bx^2$, $2b^2x^2-b^2$, $10-2bx^2$, b^2x^2-20, and $3bx^2+b^2$. *Ans.* $10b^2-2bx^2+3b^2x^2-10$.

18. Add $9bc^3-18ac^2$, $15bc^3+ac$, $9ac^2-24bc^3$, and $9ac^2-2$. *Ans.* $ac-2$.

19. Add $6m^2+2am+1$, $6am-2m^2+4$, $2m^2-8am+7$, and $3m^2-1$. *Ans.* $9m^2+11$.

20. Add $5x^4-3x^3+4x^2-2x+10$, $7x^4+2x^3+2x^2+5x+2$, and x^3-3x. *Ans.* $12x^4+6x^2+12$.

21. Add $3x^3y^2-5x^2y^3-x^2y-xy^2+5xy$, $7x^2y^3-4x^2y+2x^3y^2+2xy^2+xy$, and $x^3y^2-xy^2-2x^2y^3+5x^2y+2xy$. *Ans.* $6x^3y^2+8xy$.

22. Add $5a+3\sqrt{m^2-1}+4$, $7a-\sqrt{m^2-1}-5$, $3a-5\sqrt{m^2-1}-8$, and $2a+2\sqrt{m^2-1}+2$. *Ans.* $17a-\sqrt{m^2-1}-7$.

23. What is the sum of $3a^2c^{\frac{1}{2}}-2c^2a^{\frac{1}{2}}+a^{\frac{1}{2}}c^{\frac{1}{2}}$, $2a^2c^{\frac{1}{2}}+3c^2a^{\frac{1}{2}}-5a^{\frac{1}{2}}c^{\frac{1}{2}}$, and $a^2c^{\frac{1}{2}}-5c^2a^{\frac{1}{2}}+8a^{\frac{1}{2}}c^{\frac{1}{2}}$? *Ans.* $6a^2c^{\frac{1}{2}}-4c^2a^{\frac{1}{2}}+4a^{\frac{1}{2}}c^{\frac{1}{2}}$.

24. What is the sum of $9a(a-b)-4m\sqrt{m-c}$, $7m\sqrt{m-c}-6a(a-b)$, and $12m\sqrt{m-c}-8a(a-b)$? *Ans.* $15m\sqrt{m-c}-5a(a-b)$.

25. What is the sum of $a+b+c+d+m$, $a+b+c+d-m$, $a+b+c-d-m$, $a+b-c-d-m$, and $a-b-c-d-m$? *Ans.* $5a+3b+c-d-3m$.

50. The ***Unit of Addition*** is the letter or quantity whose coefficients are added, in the operation of finding the sum of two or more quantities. Thus, in the example,

$$3x+2x+4x=9x$$

x is the unit of addition. Also, in the example,

$$5\sqrt{a+c}+4\sqrt{a+c}-3\sqrt{a+c}=6\sqrt{a+c}$$

$\sqrt{a+c}$ is the unit of addition.

51. When dissimilar terms have a common literal part, this may be taken as the unit of addition. The sum of the terms will then be expressed by inclosing the sum of the coefficients in a parenthesis, and prefixing it to the common unit.

EXAMPLES FOR PRACTICE.

(1.)	(2.)	(3.)
ax^2	$17axy^2$	$(5a-b)\sqrt{x}$
bx^2	$-5axy^2$	$(2c-a)\sqrt{x}$
$-cx^2$	$2mxy^2$	$(b-c)\sqrt{x}$
$(a+b-c)x^2$.	$(12a+2m)xy^2$.	$(4a+c)\sqrt{x}$.

(4.)	(5.)
$a(x^2-y^2)$	$(a^2-3b)(m^2-1)$
$b(x^2-y^2)$	$(b^2-3a)(m^2-1)$
$-c(x^2-y^2)$	$(3a+3b)(m^2-1)$
$(a+b-c)(x^2-y^2)$.	$(a^2+b^2)(m^2-1)$.

6. Add ax, $2cx$, and $4dx$. *Ans.* $(a+2c+4d)x$.

7. Add $ay+cx$, $3ay+2cx$, and $4y+6x$.
Ans. $(4a+4)y+(3c+6)x$.

8. Add $3x+2xy$, $bx+cxy$, and $(a+b)x+2cdxy$.
Ans. $(a+2b+3)x+(2cd+c+2)xy$.

9. Add $ax+7y$, $7ax-3y$, and $-2x+4y$.
Ans. $(8a-2)x+8y$.

10. Add $(b-a)\sqrt{x}$, and $(c+2a-b)\sqrt{x}$. *Ans.* $(c+a)\sqrt{x}$.

11. Add $(a+2b)m-c\sqrt{m}$, $(2a-6c)m-3a\sqrt{m}$, $(5c-4a)m-b\sqrt{m}$, and $(2a-3b)m+4a\sqrt{m}$. *Ans.* $(a-b-c)(m+\sqrt{m})$.

12. Add $ax+y+z$, $x+ay+z$, and $x+y+az$.
Ans. $(a+2)(x+y+z)$.

SUBTRACTION.

52. ***Subtraction*** is the process of finding the difference between two quantities.

53. It is evident that 5 units of any kind or quality subtracted from 8 units of the same kind or quality, must leave 3 units of the same kind or quality. That is,

$$+8a-(+5a)=+3a.$$

Also, $$-8a-(-5a)=-3a.$$

But these remainders are the same as we shall obtain by changing the signs of the subtrahends and then *adding* the results, algebraically, to the minuends. Thus,

$$+8a-(+5a)=+8a-5a=+3a$$
$$-8a-(-5a)=-8a+5a=-3a$$

Hence, in Algebra,

Subtracting any quantity is equivalent to adding the same quantity with its sign changed.

54. This principle may be established in a more general manner as follows:

Let it be required to subtract the quantity $b-c$ from a.

OPERATION.

Minuend,	a
Subtrahend,	$b-c$
Difference,	$a-b+c$

We first subtract b from a, indicating the operation, and obtain for a result, $a-b$. But the true subtrahend is not b, but $b-c$; and, as we have subtracted a quantity too great by c, the remainder thus obtained must be too small by c; we therefore add c to the first result, and obtain the true remainder, $a-b+c$. But this result is the same as would be obtained by adding $-b+c$ to a.

55. It follows from the principle enunciated above, that any quantity is subtracted from nothing or zero, by simply changing its sign or signs. Thus,

$$0-(+a)=-a,$$
$$0-(-a)=+a,$$
$$0-(a-b)=-a+b.$$

56. From these principles and illustrations we deduce the following:

RULE.—I. *Write the subtrahend underneath the minuend, placing the similar terms together in the same column.*

II. *Conceive the signs of the subtrahend to be changed, unite the similar terms as in addition, and bring down all the remaining terms with their proper signs.*

EXAMPLES FOR PRACTICE.

(1.)
$$\begin{array}{r} 18x^2y \\ 12x^2y \\ \hline 6x^2y. \end{array}$$

(2.)
$$\begin{array}{r} 5mc^2 \\ 9mc^2 \\ \hline -4mc^2. \end{array}$$

(3.)
$$\begin{array}{r} 3a^2bc \\ -2a^2bc \\ \hline 5a^2bc. \end{array}$$

(4.)
$$\begin{array}{r} -5x^3y^2z \\ -7x^3y^2z \\ \hline 2x^3y^2z. \end{array}$$

(5.)
$$\begin{array}{r} 4a + 2x - 3c \\ a + 4x - 6c \\ \hline 3a - 2x + 3c. \end{array}$$

(6.)
$$\begin{array}{r} 3ax + 2y \\ ax - 2y \\ \hline 2ax + 4y. \end{array}$$

(7.)
$$\begin{array}{r} 7a^2x^3 - 4\sqrt{ax} - 3x^2y \\ 6a^2x^3 - 5\sqrt{ax} - 4x^2y \\ \hline a^2x^3 + \sqrt{ax} + x^2y. \end{array}$$

(8.)
$$\begin{array}{l} 4a^2x + c^2d + 4md^2 \\ \quad a^2x + cd^2 - 3md^2 \\ \hline 3a^2x + c^2d - cd^2 + 7md^2. \end{array}$$

(9.)
$$\begin{array}{l} 5m - b^2 + c \\ 2m + b - c^2 \\ \hline 3m - b^2 - b + c^2 + c. \end{array}$$

10. From $2x^2 - 3x + y^2$ subtract $a - x^2 - 4x$.
Ans. $3x^2 + x + y^2 - a$.

11. From $7a - 5c + 2$ subtract $-a + c + 2$. *Ans.* $8a-6c$.

12. From $8x^2 - 3xy + 2y^2 + c$ subtract $x^2 - 6xy + 3y^2 - 2c$.
Ans. $7x^2 + 3xy - y^2 + 3c$.

13. From $a + b$ subtract $a - b$. *Ans.* $2b$.

14. From $\frac{1}{2}x + \frac{1}{2}y$ subtract $\frac{1}{2}x - \frac{1}{2}y$. *Ans.* y.

15. From $a + b + c$ subtract $-a-b-c$. *Ans.* $2a+2b+2c$.

16. From $3a - b - 2x + 7$ take $8 - 3b + a + 4x$.
Ans. $2a + 2b - 6x - 1$.

17. From $6y^2 - 2y - 5$ take $-8y^2 - 5y + 12$.
Ans. $14y^2 + 3y - 17$.

18. From $3p + q + r - 3s$ take $q - 8r + 2s - 8$.
Ans. $3p + 9r - 5s + 8$.

19. From $13a^2 - 2ax + 9x^2$ take $5a^2 - 7ax - x^2$.
Ans. $8a^2 + 5ax + 10x^2$.

20. From $x^4 - 3x^3 + 5x^2 - 7x + 12$ take $x^4 - 4x^3 + 2x^2 - 6x + 15$.
Ans. $x^3 + 3x^2 - x - 3$.

21. From $a^5 - 3a^4c + 3a^3c^2 - 2a^2c^3 + 4ac^4 - c^5$ take $a^5 - 4a^4c + 2a^3c^2 - 5a^2c^3 + 3ac^4 - c^5$. *Ans.* $a^4c + 3a^3c^2 + 3a^2c^3 + ac^4$.

22. From $2x^4 + 28x^3 + 134x^2 - 252x + 144$ take $2x^4 + 21x^3 + 67x^2 - 63x + 84$. *Ans.* $7x^3 + 67x^2 - 189x + 60$.

23. From $x^5 + 5x^4y + 10x^3y^2 + 10x^2y^3 + 5xy^4 + y^5$ take $x^5 - 5x^4y + 10x^3y^2 - 10x^2y^3 + 5xy^4 - y^5$. *Ans.* $10x^4y + 20x^2y^3 + 2y^5$.

24. From the sum of $6x^2y - 11ax^3$ and $8x^2y + 3ax^3$, take $4x^2y - 4ax^3 + a$. *Ans.* $10x^2y - 4ax^3 - a$.

25. From the sum of $8cdx + 15a^2b - 3$ and $2cdx - 8a^2b + 24$ take the sum of $12a^2b - 3cdx - 8$ and $cdx - 4a^2b + 16$.
Ans. $12cdx - a^2b + 13$.

57. The difference of two dissimilar terms may often be conveniently expressed in a single term, as in (**51**), by taking some common letter or letters as the unit of subtraction.

EXAMPLES.

(1.)	(2.)	(3.)
$2cx$	mx^2y^2	$ax + by$
mx	$-4x^2y^2$	$cx - y$
$(2c - m)x$.	$(m + 4)x^2y^2$.	$(a - c)x + (b + 1)y$.

4. From $c^2d^2m + 4ax^2$ take $d^2m + 3ax^2$. *Ans.* $(c^2 - 1)d^2m + ax^2$.

5. From $ax + by + cz$ take $mx + ny + pz$.
Ans. $(a - m)x + (b - n)y + (c - p)z$.

6. From $ax + bx + cx$ take $x + ax + bx$. *Ans.* $(c - 1)x$.

7. From $(a + 2b + c)\sqrt{xy}$ take $(2b - c)\sqrt{xy}$.
Ans. $(a + 2c)\sqrt{xy}$.

8. From $(3a - 2m)x^3 + (5a + 2m)x^2 + (4a - m)x$ take $(a - m)x^3 - (2a + m)x^2 + (2a - 3m)x$.
Ans. $(2a - m)x^3 + (7a + 3m)x^2 + (2a + 2m)x$.

9. From $1+2az^2+3a^2z^4+4a^3z^6+5a^4z^8$ take $z^2+2az^4+3a^2z^6+4a^3z^8$.

Ans. $1+(2a-1)z^2+(3a^2-2a)z^4+(4a^3-3a^2)z^6+(5a^4-4a^3)z^8$.

USE OF THE PARENTHESIS.

58. The term, *parenthesis*, will be employed hereafter as a general name to designate the various signs of aggregation employed in algebraic operations. The following rules respecting the use of the parenthesis should be thoroughly considered by the learner, if he would acquire facility in algebraic transformations.

59. From the definition of the signs of aggregation (**17**), we understand that if the *plus* sign occurs before a parenthesis, all the terms enclosed are to be added, which does not require that the signs of the terms be changed ; but if the *minus* sign occurs before a parenthesis, all the terms enclosed are to be subtracted, which requires that the signs of all the terms be changed. Hence,

1. *A parenthesis preceded by the plus sign may be removed, and the enclosed terms written with their proper signs.* Thus,

$$a-b+(c-d+e)=a-b+c-d+e.$$

2. Conversely : *Any number of terms, with their proper signs, may be enclosed by a parenthesis, and the plus sign written before the whole.* Thus,

$$a-b+c-d+e=a+(-b+c-d+e).$$

3. *A parenthesis preceded by the minus sign may be removed, provided the signs of all the enclosed terms be changed.* Thus,

$$a-(b-c+d-e)=a-b+c-d+e.$$

4. Conversely : *Any number of terms may be enclosed by a parenthesis, preceded by the minus sign, provided the sign of every term thus enclosed be changed.* Thus,

$$a-b+c-d+e=a-b+c-(d-e).$$

60. When two or more parentheses are used in the same expression, they may be removed successively by the above rules. Thus,

$$a-\{b-c-(d-e)\}=a-\{b-c-d+e\}=a-b+c+d-e.$$

Or, in a different order,

$$a-\{b-c-(d-e)\}=a-b+c+(d-e)=a-b+c+d-e.$$

EXAMPLES FOR PRACTICE.

61. Remove the parentheses from the following expressions, and reduce the results:

1. $3a + (2b^2 - a - d + m)$. *Ans.* $2a + 2b^2 - d + m$.

2. $4x^2 - y - (3x - 7y + 5) + 2x$. *Ans.* $4x^2 + 6y - x - 5$.

3. $a + 2c - (4c - 3a + 2m^2)$. *Ans.* $4a - 2c - 2m^2$.

4. $4x^3 - 2x^2 - [x^3 - (2x^2 + 5x - 7) - 6x + 1]$.
Ans. $3x^3 + 11x - 8$.

5. $a + 2m - \{c + x - [a - m - (c - 2x)]\}$.
Ans. $2a + m - 2c + x$.

6. $3x^2 - 4x - am - \{x^2 - x - [3am - (2x + 2am) + 2x^2] - 5am\}$.
Ans. $4x^2 - 5x + 5am$.

7. $3a - \{2m^2 + [5c - 9a - (3a + m^2)] + 6a - (m^2 + 5c)\}$.
Ans. $9a$.

8. $x^2 - \{5mc^2 - [x^2 - (3c - 3mc^2) + 3c - (x^2 - 3mc^2 - c)]\}$.
Ans. $x^2 + c$.

9. $m^2 - m - 1 - \{m^2 - 2m - 2 - [m^2 - 3m - 3 - (m^2 - 4m - 4)]\}$.
Ans. $2m + 2$.

10. $5z^3 - 3z^2 + 4z - 1 - [2z^3 - (3z^2 - 2z + 1) - z^3 + z]$.
Ans. $4z^3 + z$.

11. $4c^3 - 2c^2 + c + 1 - (3c^3 - c^2 - c - 7) - (c^3 - 4c^2 + 8)$.
Ans. $3c^2$.

12. $3a^2b - 4cd - (3cd - 2a^2b) - [a^3 + c - (5cd + 3a^2b) + (3a^3 + 2cd) + a^3]$. *Ans.* $8a^2b - 4cd - 5a^3 - c$.

13. $-\Big(4a^2m + 3m^2d - (7m^2d - 9a^2m - n) - \{5n - [m^2d - (2n + a^2m) + 3an^2] - 5a^2m\} - 12a^2m\Big)$. *Ans.* $3m^2d + 6n - 5a^2m - 3an^2$.

62. In Algebra, addition does not necessarily imply augmentation, nor does subtraction always imply diminution, in an arithmetical sense.

We have seen that one quantity is added to another by annexing it with its proper sign; but a quantity is subtracted from another by annexing it with its sign changed. Hence,

1st. Adding a positive quantity has the same effect as subtracting a negative quantity; and adding a negative quantity has the same effect as subtracting a positive quantity.

2d. If to any given quantity a positive quantity be added, the result will be greater than the given quantity; but if a negative quantity be added, the result will be less than the given quantity.

3d. If from any given quantity a positive quantity be subtracted, the result will be less than the given quantity; but if a negative quantity be subtracted, the result will be greater than the given quantity.

63. Let $-a$ denote any negative quantity. Add $-b$ to this quantity, and subtract $+b$ from it; and we have

$$-a+(-b)=-a-b,$$
$$-a-(+b)=-a-b,$$

But according to the last two propositions, the result, $-a-b$, should be less than the given quantity, $-a$. That is,

$$-a-b<-a.$$

Now, the quantity, $-a-b$, contains a greater number of units than $-a$. These cases, however, are not exceptions to the laws enunciated above; for in an *algebraic sense,* the less of two negative quantities is that one which contains the greater number of units. (See **197.**)

64. If a represent the greater of the two numbers, and b the less, then $a+b$ is their sum and $a-b$ their difference; and the sum and difference may be combined in two ways, as follows:

1st;	To	$a+b$	2d;	From	$a+b$
	Add	$a-b$		Subtract	$a-b$
		$2a.$			$2b.$

Hence,

1. *If the difference of two numbers be added to their sum, the result will be twice the greater number.*

2. *If the difference of two numbers be subtracted from their sum, the result will be twice the less number.*

MULTIPLICATION.

65. ***Multiplication*** is the process of taking one quantity as many times as there are units in another.

66. In order to establish general rules for multiplication, we must first consider the simple case of multiplying one monomial by another; and we will investigate, first, *The law of coefficients;* second, *The law of exponents;* third, *The law of signs.*

1ST. THE LAW OF COEFFICIENTS.

Let it be required to multiply $5a$ by $3b$. Since it is immaterial in what order the factors are taken, we may proceed thus: $5 \times 3 = 15$; $a \times b = ab$; and $15 \times ab = 15ab$. Or $5a \times 3b = 15ab$. Hence,

The coefficient of the product is equal to the product of the coefficients of the multiplicand and multiplier.

2D. THE LAW OF EXPONENTS.

Let it be required to multiply a^4b^3 by a^3b^2. Since $a^4b^3 = aaaabbb$, and $a^3b^2 = aaabb$, we have

$$a^4b^3 \times a^3b^2 = aaaabbbaaabb = a^7b^5.$$

Hence,

The exponent of any letter in the product is equal to the sum of the exponents of this letter in the multiplicand and multiplier.

3D. THE LAW OF SIGNS.

In Arithmetic, multiplication is restricted to the simple process of repeating a number; and the only idea attached to a multiplier is, that it shows *how many times* the multiplicand is to be taken. In Algebra, however, a multiplier may be affected by either the plus or the minus sign; and it is necessary to consider how the *sign* of the multiplier modifies its signification.

For this purpose, suppose it were required to multiply any quantity, as a, by $c - d$. Now it is evident that a taken c minus d times, is the same as a taken c times, diminished by a taken d times; or $a \times (c - d) = ac - ad$. In the first term of this result, a is taken c times positively, or $a + a + a + a$, etc., to c repetitions; and this is the product of a by $+ c$. In the second

term, a is taken d times negatively, or $-a-a-a-$, etc., to d repetitions; and this is the product of a by $-d$. Hence we conclude that the signs, $+$ and $-$, when prefixed to a multiplier, must be interpreted as follows:

The plus sign before a multiplier shows that the multiplicand is to be successively added; and the minus sign before a multiplier shows that the multiplicand is to be successively subtracted.

To exhibit the law which governs the sign of a product, according to this principle, we present the four cases which involve all the variations of signs. It will be observed that according to the above interpretation, the multiplicand is to be repeated with its proper sign when the multiplier is positive, but with its sign changed when the multiplier is negative. We shall therefore have the following results:

1. $+a \times (+b) = +a+a+a+\text{etc.} = +ab.$
2. $+a \times (-b) = -a-a-a-\text{etc.} = -ab.$
3. $-a \times (+b) = -a-a-a-\text{etc.} = -ab.$
4. $-a \times (-b) = +a+a+a+\text{etc.} = +ab.$

Comparing the first result with the fourth, and the second with the third, we observe that

When the two factors have like signs, the product is positive; and when the two factors have unlike signs, the product is negative.

67. This law applied in the case of two or more negative factors gives the following results:

$$(-a)\times(-b) = +ab$$
$$(-a)\times(-b)\times(-c) = (+ab)\times(-c) = -abc$$
$$(-a)\times(-b)\times(-c)\times(-d) = (-abc)\times(-d) = +abcd$$
$$(-a)\times(-b)\times(-c)\times(-d)\times(-e) = (+abcd)\times(-e) = -abcde$$

Hence the general truth:

The product of an even number of negative factors is positive; and the product of an odd number of negative factors is negative.

CASE I.

68. When both factors are monomials.

From the principles already established we derive the following

RULE.—I. *Multiply the coefficients of the two terms together for the coefficient of the product.*

II. *Write all the letters of both terms for the literal part, giving each an exponent equal to the sum of its exponents in the two terms.*

III. *If the signs of the two terms are alike, prefix the plus sign to the product; if unlike, prefix the minus sign.*

EXAMPLES FOR PRACTICE.

(1.)	(2.)	(3.)	(4.)
$7x^2y$	a^3cm^2	$-5c^4m^2$	$-4x^2y^3z^4$
$5xy^3$	$-6ac^4d$	$3c^2d^2$	$-2x^4yz$
$35x^3y^4.$	$-6a^4c^5m^2d.$	$-15c^6m^2d^2.$	$8x^6y^4z^5.$

5. Multiply $17a^3b^2c^3$ by $7ac$.
6. Multiply $11a^5b^2c$ by $10a^5b^8c^9$. *Ans.* $110a^{10}b^{10}c^{10}$.
7. Multiply $117ab^2c^3x$ by $2a^3b^2c$.
8. Multiply $7x^2yz^4$ by $-4xyz$. *Ans.* $-28x^3y^2z^5$.
9. Multiply $-12cd^2m^4$ by $10c^4$. *Ans.* $-120c^5d^2m^4$.
10. Multiply $-15a^3bx^3y$ by $-3ab^2y$. *Ans.* $45a^4b^3x^3y^2$.
11. Multiply a^m by a^n. *Ans.* a^{m+n}.
12. Multiply x^my by xy^m. *Ans.* $x^{m+1}y^{m+1}$.
13. Multiply $4a^mb^nc$ by $-6a^2b^3c$. *Ans.* $-24a^{m+2}b^{n+3}c^2$.
14. Multiply $3x^cy^m$ by $2x^{2c}y^{3m}$. *Ans.* $6x^{3c}y^{4m}$.
15. What is the continued product of $3x$, $2x^2y$, and $7x^2y^3z$? *Ans.* $42x^5y^4z$.
16. What is the continued product of $5a^2b$, ab^3, $3a^2c$, and $-5abc$? *Ans.* $-75a^6b^5c^2$.
17. What is the continued product of $7xy$, $-2x^2$, $3x^2y$, $-xy^3$, and xy^2?
18. What is the continued product of $-3c^2dm$, $-2cd^2m$, and $-5cdm^2$? *Ans.* $-30c^4d^4m^4$.
19. What is the continued product of $-a$, $-ab$, $-abc$, $-abcd$, $-abcdh$, and $-abcdhm$? *Ans.* $a^6b^5c^4d^3h^2m$.
20. Multiply $2(x+y)$ by $4a^2(x+y)$. *Ans.* $8a^2(x+y)^2$.
21. Multiply $4m^2(x-z)^2$ by $-(z-x)$. *Ans.* $-4m^2(x-z)^3$.
22. Multiply $(a-c)^{m+1}$ by $(a-c)^{m-1}$. *Ans.* $(a-c)^{2m}$.

CASE II.

69. When one or both of the factors are polynomials.

1. Multiply $x-y+z$ by $a+b-c$.

OPERATION

$$
\begin{array}{lrrrrrrrrr}
 & x & -y & +z & & & & & & \\
 & a & +b & -c & & & & & & \\
\hline
\text{Product by } a, & ax & -ay & +az & & & & & & \\
\text{Product by } b, & & & & +bx & -by & +bz & & & \\
\text{Product by } -c. & & & & & & & -cx & +cy & -cz \\
\hline
\text{Entire Product,} & ax & -ay & +az & +bx & -by & +bz & -cx & +cy & -cz.
\end{array}
$$

Hence the following general

RULE.—*Multiply all the terms of the multiplicand by each term of the multiplier, and add the partial products.*

EXAMPLES FOR PRACTICE.

(1.)

$$
\begin{array}{l}
3a - 2bc \\
2a^2 \\
\hline
6a^3 - 4a^2bc.
\end{array}
$$

(2.)

$$
\begin{array}{l}
5x^2y + 2xy^2 \\
3xy \\
\hline
15x^3y^2 + 6x^2y^3.
\end{array}
$$

(3.)

$$
\begin{array}{l}
4a^2m - 3cd^2 \\
- 3ac^2 \\
\hline
- 12a^3c^2m + 9ac^3d^2.
\end{array}
$$

(4.)

$$
\begin{array}{rrr}
3x & +2y & \\
4x & -5y & \\
\hline
12x^2 & +8xy & \\
 & -15xy & -10y^2 \\
\hline
12x^2 & -7xy & -10y^2.
\end{array}
$$

(5.)

$$
\begin{array}{rrrr}
2x^2 + xy - 2y^2 & & & \\
3x - 3y & & & \\
\hline
6x^3 & +3x^2y & -6xy^2 & \\
 & -6x^2y & -3xy^2 & +6y^3 \\
\hline
6x^3 & -3x^2y & -9xy^2 & +6y^3.
\end{array}
$$

6. Multiply $3a^3x^2 - y^2z + z^2$ by $2az^2$.

Ans. $6a^4x^2z^2 - 2ay^2z^3 + 2az^4$.

7. Multiply $x^4 - 3x^3 + 2x^2 - 5x + 3$ by $3x^2$.

Ans. $3x^6 - 9x^5 + 6x^4 - 15x^3 + 9x^2$.

8. Multiply $a^3c^2 - 3a^2c^3 + a^2c - ac^2 + a - c + 1$ by ac.

9. Multiply $2ax - 3x$ by $2x + 4y$.
Ans. $4ax^2 + 8axy - 6x^2 - 12xy$.

10. Multiply $3a^2 - 2ab - b^2$ by $2a - 4b$.
Ans. $6a^3 - 16a^2b + 6ab^2 + 4b^3$.

11. Multiply $x^2 - xy + y^2$ by $x + y$. *Ans.* $x^3 + y^3$.

12. Multiply $a^2-3ac+c^2$ by $a-c$. *Ans.* $a^3-4a^2c+4ac^2-c^3$.

13. Multiply $2x^2 - 3x + 2$ by $x - 8$.
Ans. $2x^3 - 19x^2 + 26x - 16$.

14. Multiply $a^3 + 2a^2b + 2ab^2 + b^3$ by $a^3 - 2a^2b + 2ab^2 - b^3$.
Ans. $a^6 - b^6$.

15. Multiply $a^m + b^m$ by $a^n + b^n$.
Ans. $a^{m+n} + a^nb^m + a^mb^n + b^{m+n}$.

16. Multiply $4x^3+8x^2+16x+32$ by $3x-6$. *Ans.* $12x^4-192$.

17. Multiply $a^3 + a^2b + ab^2 + b^3$ by $a - b$. *Ans.* $a^4 - b^4$.

NOTE.—The product of two or more polynomials may be indicated by inclosing each in a parenthesis, and writing them one after another, with or without the sign, ×, between the parentheses. Such an expression is said to be *expanded*, when the indicated multiplication has been actually performed.

18. Expand $(a + m)(a + d)$. *Ans.* $a^2 + am + ad + dm$.

19. Expand $(a+2m-1)(a + 1)$. *Ans.* $a^2+2am+2m-1$.

20. Expand $(z^3 + 4z^2 + 5z - 24)(z^2 - 4z + 11)$.
Ans. $z^5 + 151z - 264$.

21. Expand $(a^3 - 4a^2 + 11a - 24)(a^2 + 4a + 5)$.
Ans. $a^5 - 41a - 120$.

22. Expand $(m - 3)(m - 1)(m + 1)(m + 3)$.
Ans. $m^4 - 10m^2 + 9$.

23. Expand $(x^3 - 2x^2 + 3x - 4)(4x^3 + 3x^2 + 2x + 1)$.
Ans. $4x^6 - 5x^5 + 8x^4 - 10x^3 - 8x^2 - 5x - 4$.

24. Expand $(y^4 + 2y^3 + y^2 - 4y - 11)(y^2 - 2y + 3)$.
Ans. $y^6 + 10y - 33$.

25. Expand $(c^2 - c + 1)(c^2 + c + 1)(c^4 - c^2 + 1)$.
Ans. $c^8 + c^4 + 1$.

26. Expand $(x^5 - 5x^4 + 13x^3 - x^2 - x + 2)(x^2 - 2x - 2)$.
Ans. $x^7 - 7x^6 + 21x^5 - 17x^4 - 25x^3 + 6x^2 - 2x - 4$.

27. Expand $(16x^4 - 8x^3 + 4x^2 - 2x + 1)(2x + 1)$.
Ans. $32x^5 + 1$.

FORMULAS AND GENERAL PRINCIPLES.

70. A ***Formula*** is the algebraic expression of a general truth or principle.

The following formulas are useful, as furnishing rules for obtaining the products of certain binomial factors.

If a and b represent any two quantities whatever, then

$$a + b = \text{their sum, and}$$
$$a - b = \text{their difference;}$$

and we have, after performing the indicated operations, the results which follow:

I. $$(a + b)^2 = (a + b)(a + b) = a^2 + 2ab + b^2;$$

hence,

The square of the sum of two quantities is equal to the square of the first, plus twice the product of the first and second, plus the square of the second.

II. $$(a - b)^2 = (a - b)(a - b) = a^2 - 2ab + b^2;$$

hence,

The square of the difference of two quantities is equal to the square of the first, minus twice the product of the first and second, plus the square of the second.

III. $$(a + b)(a - b) = a^2 - b^2;$$

hence,

The product of the sum and difference of two quantities is equal to the difference of their squares.

By the aid of these formulas we are enabled to write the square of any binomial, or the product of the sum and difference of any two quantities, without formal multiplication.

EXAMPLES FOR PRACTICE.

1. What is the square of $3a + 2ab$?

The square of the first term is $9a^2$, twice the product of the two terms is $12a^2b$, and the square of the second term is $4a^2b^2$; hence, by the first formula,

$$(3a + 2ab)^2 = 9a^2 + 12a^2b + 4a^2b^2, \quad \textit{Ans.}$$

2. What is the square of $2x^2 - 5$?

The square of the first term is $4x^4$, twice the product of the two terms is $20x^2$, and the square of the second term is 25; hence by the second formula,

$$(2x^2 - 5)^2 = 4x^4 - 20x^2 + 25, \textit{ Ans.}$$

3. What is the product of $5x + y^2$ and $5x - y^2$?

The square of $5x$ is $25x^2$, and the square of y^2 is y^4; hence by the third formula,

$$(5x + y^2)(5x - y^2) = 25x^2 - y^4, \textit{ Ans.}$$

4. What is the square of $c + m$? *Ans.* $c^2 + 2cm + m^2$.
5. What is the square of $x - y$? *Ans.* $x^2 - 2xy + y^2$.
6. What is the product of $x + y$ and $x - y$? *Ans.* $x^2 - y^2$.
7. What is the square of $3x^2 + 4y$? *Ans.* $9x^4 + 24x^2y + 16y^2$.
8. What is the square of $5c^3 - 2cd$?
Ans. $25c^6 - 20c^4d + 4c^2d^2$.
9. What is the product of $4z^2 + 3yz$ and $4z^2 - 3yz$?
Ans. $16z^4 - 9y^2z^2$.
10. What is the square of $3a^2x + 2ay$?
Ans. $9a^4x^2 + 12a^3xy + 4a^2y^2$.
11. What is the square of $x + 1$? *Ans.* $x^2 + 2x + 1$.
12. What is the square of $2z^2 - 1$? *Ans.* $4z^4 - 4z^2 + 1$.
13. What is the product of $m + 1$ and $m - 1$? *Ans.* $m^2 - 1$.
14. What is the square of $z^2 - 30$? *Ans.* $z^4 - 60z^2 + 900$.
15. What is the product of $3a^2b + d^3$ and $3a^2b - d^3$?
Ans. $9a^4b^2 - d^6$.
16. What is the square of $x - \frac{1}{2}y$? *Ans.* $x^2 - xy + \frac{1}{4}y^2$.
17. What is the square of $2c + \frac{1}{2}$? *Ans.* $4c^2 + 2c + \frac{1}{4}$.
18. What is the square of $x^m + y^n$? *Ans.* $x^{2m} + 2x^my^n + y^{2n}$.
19. What is the product of $x^m + y^n$ and $x^m - y^n$?
Ans. $x^{2m} - y^{2n}$.

71. The binomial square occurs so frequently in algebraic operations, that it is important for the student to be perfectly familiar with its form. The higher powers of any binomial may be obtained by actual multiplication. The 3d, 4th, and 5th powers, however, may sometimes be easily written, without actual multiplication, by means of the formulas which follow:

1. $(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3.$
2. $(a-b)^3 = a^3 - 3a^2b + 3ab^2 - b^3.$
3. $(a+b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4.$
4. $(a-b)^4 = a^4 - 4a^3b + 6a^2b^2 - 4ab^3 + b^4.$
5. $(a+b)^5 = a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5.$
6. $(a-b)^5 = a^5 - 5a^4b + 10a^3b^2 - 10a^2b^3 + 5ab^4 - b^5.$

Let the pupil verify the above by actual multiplication.

72. A polynomial is said to be *arranged* according to the descending powers of any letter, when the terms are so placed that the exponents of this letter diminish from left to right throughout all the terms that contain it. Thus, the polynomial

$$x^5 - 4x^4 + 2x^2 - x + 7$$

is arranged according to the descending powers of x.

73. A polynomial is said to be arranged according to the ascending powers of any letter, when the terms are so placed that the exponents of this letter increase from left to right throughout the terms that contain it. Thus, the polynomial

$$d - ax + cx^3 - bx^5$$

is arranged according to the ascending powers of x.

74. A term or quantity is said to be *independent* of any letter, when it does not contain that letter.

75. The product of two polynomials has certain special properties, which may be stated as follows:

1. If both polynomials are arranged according to the descending powers of the same letter, then the first term obtained in the partial products will contain a higher power of this letter than any of the other terms; and as this term cannot be reduced with any of the others, it will form the first term of the entire product.

2. If both polynomials are arranged according to the ascending powers of the same letter, then the last term obtained in the partial products will contain a higher power of this letter than any of the other terms; and as this term cannot be reduced with any of the others, it will form the last term of the entire product.

3. If both polynomials are homogeneous, then the product will be homogeneous; and the degree of any term will be expressed by the sum of the indices denoting the degrees of its two factors.

DIVISION.

76. ***Division*** is the process of finding how many times one quantity, called the *divisor,* is contained in another quantity, called the *dividend;* the result of division is called the *quotient.*

It follows, therefore, that the quotient must be a quantity which multiplied by the divisor, will produce the dividend. Thus, reversing the process of multiplication, we have,

$$abc \div a = bc, \quad \text{because} \quad bc \times a = abc.$$

77. It was shown in the multiplication of monomials (**66**), that the coefficient of the product is found by multiplying together the coefficients of the factors; and that the exponent of any letter in the product is found by adding together the exponents of this letter in the factors. Hence, in division,

1. *The coefficient of the quotient must be found by dividing the coefficient of the dividend by that of the divisor;* and

2. *The exponent of any letter in the quotient must be found by subtracting the exponent of this letter in the divisor from its exponent in the dividend.* Thus,

$$24a^5 \div 6a^3 = \frac{24}{6}a^{5-3} = 4a^2.$$

It was shown in multiplication (**66**), that when two factors have like signs, their product is positive; and that when two factors have unlike signs, their product is negative. In division, therefore, when the dividend is positive, the quotient must have the same sign as the divisor; and when the dividend is negative, the quotient must have the sign unlike that of the divisor. And there will be four cases, with results as follows:

1. $+ab \div (+a) = +b.$
2. $+ab \div (-a) = -b.$
3. $-ab \div (+a) = -b.$
4. $-ab \div (-a) = +b.$ Hence,

3. *If the dividend and divisor have like signs, the quotient will be positive; but if the dividend and divisor have unlike signs, the quotient will be negative.*

Case I.

78. When the divisor is a monomial.

From the principles already given we have the following

Rule.—To divide one monomial by another; —

I. *Divide the coefficient of the dividend by the coefficient of the divisor, for a new coefficient.*

II. *To this result annex the letters of the dividend, with the exponent of each diminished by the exponent of the same letter in the divisor, suppressing all letters whose exponents become zero.*

III. *If the signs of terms are alike, prefix the plus sign to the quotient; if they are unlike, prefix the minus sign.*

To divide a polynomial by a monomial:

Divide each term of the dividend separately, aad connect the quotients by their proper signs.

Note.—It may happen that the dividend will not exactly contain the divisor; in this case the division may be *indicated*, by writing the dividend above a horizontal line, and the divisor below, in the form of a fraction. The result thus obtained may be simplified, by suppressing all the factors common to the two terms; thus,

$$4x^2yz^2 \div 6x^2y^2z = \frac{4x^2yz^2}{6x^2y^2z} = \frac{2z}{3y}.$$

But as this process is essentially a case of reduction of fractions, we shall omit such examples till the subject of fractions is reached.

EXAMPLES FOR PRACTICE.

1. Divide $16ab$ by $4a$. *Ans.* $4b$.
2. Divide $21a^3c^2d$ by $7ac^2$. *Ans.* $3a^2d$.
3. Divide $-42x^5yz^4$ by $6x^3z^3$. *Ans.* $-7x^2yz$.
4. Divide $2a^6$ by a^4. *Ans.* $2a^2$.
5. Divide $-a^7$ by a^6. *Ans.* $-a$.
6. Divide $16x^3$ by $4x$. *Ans.* $4x^2$.
7. Divide $15axy^3$ by $-3ay$. *Ans.* $-5xy^2$.
8. Divide $117a^5b^4c^3$ by $3a^5bc^2$. *Ans.* $39b^3c$.
9. Divide $63a^3b^4cd^2$ by $21abcd$. *Ans.* $3a^2b^3d$.
10. Divide $63a^m$ by $7a^n$. *Ans.* $9a^{m-n}$.

11. Divide $34x^m y^n$ by $-17xy$. *Ans.* $-2x^{m-1}y^{n-1}$.
12. Divide $(a-c)^5$ by $(a-c)^3$. *Ans.* $(a-c)^2$.
13. Divide $35(x+y)^3$ by $5(x+y)$. *Ans.* $7(x+y)^2$.
14. Divide $12m^3d(c-x^2)^5$ by $3md(c-x^2)^2$.
15. Divide $3bcd+12bcx-9b^2c$ by $3bc$. *Ans.* $d+4x-3b$.
16. Divide $15a^2bc-15acx^2+5ad^2c$ by $-5ac$.
17. Divide $10x^3-15x^2-25x$ by $5x$. *Ans.* $2x^2-3x-5$.
18. Divide $15x^5-45x^4+10x^3-105x^2$ by $5x^2$.
19. Divide $a^mc-a^{m-1}c^2+a^{m-2}c^3-a^{m-3}c^4+a^{m-4}c^5$ by ac.
Ans. $a^{m-1}-a^{m-2}c+a^{m-3}c^2-a^{m-4}c^3+a^{m-5}c^4$.
20. Divide $3m^2(a-b)^2-3m(a-b)$ by $3(a-b)$.
Ans. am^2-bm^2-m.
21. Divide $7a(3m-2a)-(3m-2a)^2$ by $(3m-2a)$.
Ans. $9a-3m$.

Case II.

79. When the divisor is a polynomial.

Suppose both dividend and divisor to be arranged according to the descending powers of some letter. Then it follows, from (**75**, 1), that the first term of the dividend must be the product of the first term of the divisor by the first term of the quotient similarly arranged. We can therefore obtain this term of the quotient, by simply dividing the first term of the dividend by the first term of the divisor. The operation may then be continued in the manner of long division in Arithmetic; each remainder being treated as a new dividend, and *arranged as the first.*

1. Divide $6a^4+a^3b-20a^2b^2+17ab^3-4b^4$ by $2a^2-3ab+b^2$.

OPERATION.

$$
\begin{array}{l|l}
6a^4+a^3b-20a^2b^2+17ab^3-4b^4 & 2a^2-3ab+b^2, \text{ Divisor.} \\
\underline{6a^4-9a^3b+3a^2b^2} & \overline{3a^2+5ab-4b^2}, \text{ Quotient.} \\
\quad 10a^3b-23a^2b^2+17ab^3 & \\
\quad \underline{10a^3b-15a^2b^2+5ab^3} & \\
\qquad -8a^2b^2+12ab^3-4b^4 & \\
\qquad \underline{-8a^2b^2+12ab^3-4b^4} &
\end{array}
$$

Hence we have the following

Rule.—I. *Arrange both dividend and divisor according to the descending or ascending powers of one of the letters.*

II. *Divide the first term of the dividend by the first term of the divisor, and write the result in the quotient.*

III. *Multiply the divisor by the quotient thus found, and subtract the product from the dividend.*

IV. *Arrange the remainder for a new dividend, with which proceed as before, till the first term of the divisor is no longer contained in the first term of the remainder.*

V. *Write the final remainder, if there be any, over the divisor in the form of a fraction, and the entire result will be the quotient sought.*

EXAMPLES FOR PRACTICE.

1. Divide $a^3+3a^2x+3ax^2+x^3$ by $a+x$. *Ans.* $a^2+2ax+x^2$.
2. Divide $a^3-4a^2c+4ac^2-c^3$ by $a-c$. *Ans.* $a^2-3ac+c^2$.
3. Divide $a^3-6a^2+12a-8$ by a^2-4a+4. *Ans.* $a-2$.
4. Divide $3x^2-2x^4+x^5-x^3-2x-15$ by x^3-5-4x. *Ans.* x^2-2x+3.
5. Divide $25x^6-x^4-2x^3-8x^2$ by $5x^3-4x^2$. *Ans.* $5x^3+4x^2+3x+2$.
6. Divide $6a^4+9a^2-15a$ by $3a^2-3a$. *Ans.* $2a^2+2a+5$.
7. Divide x^6-y^6 by $x^3+2x^2y+2xy^2+y^3$. *Ans.* $x^3-2x^2y+2xy^2-y^3$.
8. Divide $ax^3-(a^2+b)x^2+b^2$ by $ax-b$. *Ans.* x^2-ax-b.
9. Divide a^4+4b^4 by $a^2-2ab+2b^2$. *Ans.* $a^2+2ab+2b^2$.
10. Divide $x^6-x^4+x^3-x^2+2x-1$ by x^2+x-1. *Ans.* $x^4-x^3+x^2-x+1$.
11. Divide $1+3x$ by $1-5x$. *Ans.* $1+8x+40x^2+200x^3+$etc.
12. Divide $1-x-x^2$ by $1+x+x^2$. *Ans.* $1-2x+2x^3-2x^4+2x^6-2x^7+2x^9-2x^{10}+$etc.
13. Divide x^6-2x^3+1 by x^2-2x+1. *Ans.* $x^4+2x^3+3x^2+2x+1$.

14. Divide $a^3 + b^3 + c^3 - 3abc$ by $a + b + c$.
Ans. $a^2 + b^2 + c^2 - bc - ac - ab$.

15. Divide $2x^7y - 5x^6y^2 - 11x^5y^3 + 5x^4y^4 - 26x^3y^5 + 7x^2y^6 - 12xy^7$ by $x^4 - 4x^3y + x^2y^2 - 3xy^3$. *Ans.* $2x^3y + 3x^2y^2 - xy^3 + 4y^4$.

16. Divide $a^5 + c^5 + a^4 + c^4 - a^4c - ac^4 - 2a^2c^2$ by $a^3 + c^3 - a^2c - ac^2$. *Ans.* $a^2 + c^2 + a + c$.

17. Divide $4x^6 - 5x^5 + 8x^4 - 10x^3 - 8x^2 - 5x - 4$ by $4x^3 + 3x^2 + 2x + 1$. *Ans.* $x^3 - 2x^2 + 3x - 4$.

18. Divide $x^5 - a^5$ by $x - a$. *Ans.* $x^4 + x^3a + x^2a^2 + xa^3 + a^4$.

19. Divide $a^3 + x^3$ by $a - x$. *Ans.* $a^2 + ax + x^2 + \dfrac{2x^3}{a - x}$.

20. Divide $x^m - xy^{m-1} - x^{m-1}y + y^m$ by $x - y$. *Ans.* $x^{m-1} - y^{m-1}$.

21. Divide $a^{c+m} - a^cb^n - a^mb^d + b^{n+d}$ by $a^m - b^n$. *Ans.* $a^c - b^d$.

22. Divide $x^{3n} - 2x^{2n}y^n - 2x^ny^{2n} + y^{3n}$ by $x^n + y^n$.
Ans. $x^{2n} - 3x^ny^n + y^{2n}$

EXACT DIVISION.

80. Division is said to be *exact* when the quotient contains no fractional part; the quotient in this case is said to be *entire.*

81. It follows from the rule of division (**78**), that the exact division of one monomial by another will be impossible under the following conditions:

1. When the coefficient of the divisor is not exactly contained in the coefficient of the dividend.

2. When a literal factor has a greater exponent in the divisor than in the dividend.

3. When a literal factor of the divisor is not found in the dividend.

82. It is also evident, from (**79**), that the exact division of one polynomial by another will be impossible,

1. When the first term of the divisor arranged with reference to any one of its letters, is not exactly contained in the first term of the dividend arranged with reference to the same letter.

2. When a remainder occurs, having no term which will exactly contain the first term of the divisor.

GENERAL RELATIONS IN DIVISION.

83. The algebraic value of a quotient depends upon the comparative values and relative signs of the dividend and divisor. Now if either the dividend or the divisor be changed with respect to its value or sign, the quotient will undergo a change, according to a certain law. As these mutual relations are frequently concerned in algebraic investigations, we present them in this place, considering first the law of change with respect to absolute value; and second, the law of change with respect to algebraic signs.

1ST. CHANGE OF VALUE.

84. In any case of exact division, the quotient is composed of those factors of the dividend which are not included among the factors of the divisor. It is evident, therefore, that if we introduce a new factor into the dividend, the divisor remaining the same, we shall introduce the same factor into the quotient; and if we exclude a factor from the dividend, the divisor remaining the same, we shall exclude this factor from the quotient.

Again, if we introduce a factor into the divisor, we shall exclude it from the quotient; and if we exclude a factor from the divisor, we shall introduce it into the quotient,—the dividend remaining the same in both cases.

Hence we have the following general principles:

I. *Multiplying the dividend multiplies the quotient, and dividing the dividend divides the quotient.*

II. *Multiplying the divisor divides the quotient, and dividing the divisor multiplies the quotient.*

III. *Multiplying or dividing both dividend and divisor by the same quantity does not change the quotient.*

2D. CHANGE OF SIGNS.

85. To show in what manner the *sign* of the quotient is affected by changing the sign of dividend or divisor, we observe that two signs can have only three relations, as follows:

$$\begin{array}{cc} + & + \\ + & - \\ - & - \end{array}$$

Now if *one* of the signs, only, in any of these couplets, be changed, the *relation* of the signs in that couplet will be changed, either from like to unlike, or from unlike to like; but if *both* of the signs in any couplet be changed, their relation will not be altered. Hence,

I. *Changing the sign of either dividend or divisor, changes the sign of the quotient.*

II. *Changing the signs of both dividend and divisor, does not alter the sign of the quotient.*

RECIPROCALS, ZERO POWERS, AND NEGATIVE EXPONENTS.

86. The ***Reciprocal*** of a quantity is the quotient obtained by dividing unity by that quantity. Thus, $\frac{1}{x}$ is the reciprocal of x; $\frac{1}{a-c}$ is the reciprocal of $a-c$.

87. In dividing any power of a quantity by any other power of the same quantity, we subtract the exponent of the divisor from the exponent of the dividend, to obtain the exponent of the quotient. Thus,

$$a^5 \div a^2 = a^{5-2} = a^3; \quad a^7 \div a^3 = a^{7-3} = a^4.$$

And in general, we have

$$\frac{a^m}{a^n} = a^{m-n}.$$

If in this expression $n = m$, the exponent of the quotient will be 0; and if $n > m$, the exponent of the quotient will be negative. Thus,

$$\frac{a^2}{a^2} = a^0; \ \frac{a^2}{a^3} = a^{2-3} = a^{-1}; \ \frac{a^2}{a^4} = a^{2-4} = a^{-2}, \text{ etc.}$$

88. It has been found useful for certain purposes in Algebra, to employ the notation, a^0, a^{-1}, a^{-2}, a^{-3}, etc. We will therefore proceed to find the meaning of such expressions.

Let a represent any quantity, and m the exponent of any power. Then by the rule of division,

$$\frac{a^m}{a^m} = a^{m-m} = a^0.$$

But the quotient obtained by dividing any quantity by itself must be equal to 1. That is,

$$\frac{a^m}{a^m} = 1.$$

Therefore, by Ax. 7, we have

$$a^0 = 1.$$

Hence,

1. *Any quantity having a cipher for its exponent is equal to unity.*

Again by the rule of division, we have

$$\frac{a^0}{a^m} = a^{0-m} = a^{-m}.$$

But we have already shown that $a^0=1$. Substituting this value for the dividend, we obtain the quotient in another form; thus,

$$\frac{a^0}{a^m} = \frac{1}{a^m}.$$

Therefore, by Ax. 7, we have

$$a^{-m} = \frac{1}{a^m}.$$

Hence,

2. *Any quantity having a negative exponent is equal to the reciprocal of that quantity with an equal positive exponent.*

DIVISIBILITY OF QUANTITIES IN THE FORM OF $a^m \pm b^m$.

89. There are certain cases of exact division of quantities in the form of $a^m + b^m$ or $a^m - b^m$, which have important applications. These may be exhibited in four general problems, as follows:

1. Divide $a^m + b^m$ by $a + b$.

Commencing the division, we have

$$\begin{array}{lrrr|l}
 & a^m & + & b^m & a+b \\
 & a^m & + a^{m-1}b & & a^{m-1} - a^{m-2}b \\
\hline
\text{1st rem.} & -a^{m-1}b & + & b^m & = -b(a^{m-1} - b^{m-1}) \\
 & -a^{m-1}b & - a^{m-2}b^2 & & \\
\hline
\text{2d rem.} & +a^{m-2}b^2 & + & b^m & = +b^2(a^{m-2} + b^{m-2}).
\end{array}$$

Now if this operation be continued, it is evident from the form which the first and second remainders assume, that when the exponent, m, is an odd number, the m^{th} remainder will be

$$-b^m(a^{m-m}-b^{m-m}) = -b^m(a^0-b^0) = -b^m(1-1) = 0;$$

the division will therefore be exact. But if m be even, the m^{th} remainder will be

$$+b^m(a^{m-m}+b^{m-m}) = +b^m(a^0+b^0) = +b^m(1+1) = +2b^m;$$

hence, the division will not be exact. Hence,

The sum of the same powers of two quantities is divisible by the sum of the quantities, if the exponent is odd, but not otherwise.

2. Divide $a^m + b^m$ by $a - b$.

Commencing the division, we have

$$\begin{array}{lllll}
 & a^m & + & b^m & \big| \; a-b \\
 & a^m & -a^{m-1}b & & \big| \; a^{m-1}+a^{m-2}b \\
\text{1st rem.} & +a^{m-1}b & + & b^m & = +b(a^{m-1}+b^{m-1}) \\
 & +a^{m-1}b & -a^{m-2}b^2 & & \\
\text{2d rem.} & +a^{m-2}b^2 & + & b^m & = +b^2(a^{m-2}+b^{m-2}).
\end{array}$$

If this operation be continued, it is evident that whether m be odd or even, the m^{th} remainder will be

$$+b^m(a^{m-m}+b^{m-m}) = +b^m(a^0+b^0) = +b^m(1+1) = +2b^m;$$

the division cannot, therefore, be exact. Hence,

The sum of the same powers of two quantities is not divisible by the difference of the quantities.

3. Divide $a^m - b^m$ by $a + b$.

Commencing the division, we have

$$\begin{array}{lllll}
 & a^m & - & b^m & \big| \; a+b \\
 & a^m & +a^{m-1}b & & \big| \; a^{m-1}-a^{m-2}b \\
\text{1st rem.} & -a^{m-1}b & - & b^m & = -b(a^{m-1}+b^{m-1}) \\
 & -a^{m-1}b & -a^{m-2}b^2 & & \\
\text{2d rem.} & +a^{m-2}b^2 & - & b^m & = +b^2(a^{m-2}-b^{m-2}).
\end{array}$$

If this operation be continued, then it is evident that when m is odd, the m^{th} remainder will be

$$-b^m(a^{m-m}+b^{m-m}) = -b^m(a^0+b^0) = -b^m(1+1) = -2b^m;$$

hence, the division cannot be exact. But if m be even, the m^{th} remainder will be

$$+b^m(a^{m-m}-b^{m-m}) = +b^m(a^0-b^0) = +b^m(1-1) = 0;$$

hence, the division in this case will be exact. Hence,

The difference of the same powers of two quantities is divisible by the sum of the quantities, if the exponent is even, but not otherwise.

4. Divide $a^m - b^m$ by $a - b$.

Commencing the division, we have

$$\begin{array}{lll|l}
 & a^m \quad - \quad b^m & & a - b \\
 & a^m \quad - a^{m-1}b & & a^{m-1} + a^{m-2}b \\ \hline
\text{1st rem.} & + a^{m-1}b - \quad b^m & = + b(a^{m-1} - b^{m-1}) & \\
 & + a^{m-1}b - a^{m-2}b^2 & & \\ \hline
\text{2d rem.} & + a^{m-2}b^2 - \quad b^m & = + b^2(a^{m-2} - b^{m-2}). &
\end{array}$$

If this operation be continued, then it is evident that whether m be odd or even, the m^{th} remainder will be

$$+ b^m(a^{m-m} - b^{m-m}) = + b^m(a^0 - b^0) = + b^m(1 - 1) = 0;$$

the division will therefore be exact. Hence,

The difference of the same powers of two quantities is always divisible by the difference of the quantities.

90. If we continue the division in the 1st, 3d, and 4th of the preceding problems, then in the cases of exact division, the form of the quotients will be as follows:

$$\frac{a^m+b^m}{a+b} = a^{m-1} - a^{m-2}b + a^{m-3}b^2 - a^{m-4}b^3 + \ldots - ab^{m-2} + b^{m-1} \ldots (1),$$

$$\frac{a^m-b^m}{a+b} = a^{m-1} - a^{m-2}b + a^{m-3}b^2 - a^{m-4}b^3 + \ldots + ab^{m-2} - b^{m-1} \ldots (2),$$

$$\frac{a^m-b^m}{a-b} = a^{m-1} + a^{m-2}b + a^{m-3}b^2 + a^{m-4}b^3 + \ldots + ab^{m-2} + b^{m-1} \ldots (3).$$

91. By giving particular values to m in (1), (2), and (3), we obtain the following results, which may be useful for reference:

$$\left.\begin{aligned}
\frac{a^3 + b^3}{a + b} &= a^2 - ab + b^2 \\
\frac{a^5 + b^5}{a + b} &= a^4 - a^3b + a^2b^2 - ab^3 + b^4
\end{aligned}\right\} \ldots (1),$$

$$\left.\begin{aligned}
\frac{a^2 - b^2}{a + b} &= a - b \\
\frac{a^4 - b^4}{a + b} &= a^3 - a^2b + ab^2 - b^3 \\
\frac{a^6 - b^6}{a + b} &= a^5 - a^4b + a^3b^2 - a^2b^3 + ab^4 - b^5
\end{aligned}\right\} \ldots (2),$$

$$\left.\begin{aligned}\frac{a^2-b^2}{a-b}&=a+b\\ \frac{a^3-b^3}{a-b}&=a^2+ab+b^2\\ \frac{a^4-b^4}{a-b}&=a^3+a^2b+ab^2+b^3\\ \frac{a^5-b^5}{a-b}&=a^4+a^3b+a^2b^2+ab^3+b^4\end{aligned}\right\}\ldots(3).$$

In like manner we may obtain

$$\left.\begin{aligned}\frac{x^3+1}{x+1}&=x^2-x+1\\ \frac{x^5+1}{x+1}&=x^4-x^3+x^2-x+1\end{aligned}\right\}\ldots(4),$$

$$\left.\begin{aligned}\frac{x^2-1}{x+1}&=x-1\\ \frac{x^4-1}{x+1}&=x^3-x^2+x-1\\ \frac{x^6-1}{x+1}&=x^5-x^4+x^3-x^2+x-1\end{aligned}\right\}\ldots(5),$$

$$\left.\begin{aligned}\frac{x^2-1}{x-1}&=x+1\\ \frac{x^3-1}{x-1}&=x^2+x+1\\ \frac{x^4-1}{x-1}&=x^3+x^2+x+1\\ \frac{x^5-1}{x-1}&=x^4+x^3+x^2+x+1\end{aligned}\right\}\ldots(6).$$

FACTORING.

92. The ***Factors*** of a quantity are those quantities which, being multiplied together, will produce the given quantity.

93. A ***Prime Factor*** is one which cannot be produced by the multiplication of two or more factors; it is therefore divisible only by itself and unity.

94. An algebraic expression may be factored by inspection, by trial, or by its law of formation.

To express the prime factors of a monomial, we have only to factor the coefficient, and repeat each letter as many times as there are units in its exponent. Thus,

$$15a^3x^2y = 3 \times 5 \times aaaxxy.$$

95. The following remarks will aid in factoring polynomials:

1st. If all the terms of a polynomial have a common factor, the quantity may be factored by writing the other factors of each term within a parenthesis, and the common factor without. Thus,

$$2a^2x^3-6a^2x^2+4a^2x-10a^3 = 2a^2(x^3-3x^2+2x-5a).$$

2d. If two of the terms of a trinomial are perfect squares, and the other term is twice the product of the square roots of the squares, the trinomial will be the square of the sum or difference of these roots (**70**, I and II), and may be factored accordingly. Thus, in the trinomial, $4a^4-20a^2b+25b^2$, the two terms, $4a^4$ and $25b^2$, are the squares of $2a^2$ and $5b$ respectively, and the other term, $20a^2b$, is equal to $2 \times 2a^2 \times 5b$; hence,

$$4a^4 - 20a^2b + 25b^2 = (2a^2 - 5b)(2a^2 - 5b).$$

3d. If a binomial consists of two squares connected by the minus sign, it must be equal to the product of the sum and difference of the square roots of the terms (**70**, III). Thus,

$$9x^2 - y^2 = (3x + y)(3x - y).$$

4th. Quantities in the form of $a^m \pm b^m$ may be factored by reference to the principles and formulas relating to these quantities. Thus,

$$a^3 + b^3 = (a + b)(a^2 - ab + b^2).$$

NOTE.—It may happen that when there is no factor common to all the terms, a portion of the polynomial may be factored.

EXAMPLES FOR PRACTICE.

1. Factor $a^3b + a^2b^2 + a^2bc$. *Ans.* $a^2b(a + b + c)$.
2. Factor $3x^2y^2 - 3x^4y^2 + 3x^2y^4 - 6x^3y^3$.
   Ans. $3x^2y^2(1 - x^2 + y^2 - 2xy)$.
3. Factor $5a^3bc^2 - 15a^2b^2c^3 - 5a^2bc^2d$.
   Ans. $5a^2bc^2(a - 3bc - d)$.

4. Factor $a^2 + c^2x + cmx$. *Ans.* $a^2 + c(c + m)x$.

5. Factor $x^3 - x^2y + xy^2 - y^3$.

6. Factor $a^4b^2 + 2a^3b^3 + a^2b^4$. *Ans.* $a^2b^2(a + b)(a + b)$.

7. Arrange $(x^2 - x)a + (x^2 + x)(3b - c) - q$ according to the powers of x. *Ans.* $(a + 3b - c)x^2 - (a - 3b + c)x - q$.

8. Factor $a^5m - 9am^3$. *Ans.* $am(a^2 - 3m)(a^2 + 3m)$.

9. Factor $8a^3 - x^3$. *Ans.* $(4a^2 + 2ax + x^2)(2a - x)$.

10. Factor $y^5 + 243$. *Ans.* $(y^4 - 3y^3 + 9y^2 - 27y + 81)(y + 3)$.

11. Find the factors of $x^6 - y^6$.

Ans. $(x^2 + xy + y^2)(x^2 - xy + y^2)(x + y)(x - y)$.

12. Find the factors of $a^3 - ab^2 + 2abc - ac^2$.

Ans. $a(a + b - c)(a - b + c)$.

SUBSTITUTION.

96. ***Substitution,*** in Algebra, is the process of putting one quantity for another, in any given expression.

1. Substitute $y - 1$ for x, in $x^3 + x^2 - 5x - 3$.

OPERATION.

$$\begin{array}{rcl} x^3 = & (y-1)^3 = & y^3 - 3y^2 + 3y - 1 \\ x^2 = & (y-1)^2 = & y^2 - 2y + 1 \\ -5x = & -5(y-1) = & -5y + 5 \\ -3 = & & -3 \\ \hline \end{array}$$

$$x^3 + x^2 - 5x - 3 = y^3 - 2y^2 - 4y + 2, \textit{ Ans.}$$

Hence, for substitution we have the following

RULE.—*Perform the same operations upon the substituted quantity as the expression requires to be performed upon the quantity for which the substitution is made.*

EXAMPLES FOR PRACTICE.

1. Substitute $a-b$ for a in a^2+ab+b^2. *Ans.* a^2-ab+b^2.

2. Substitute $x+2$ for a in a^2-2a+1. *Ans.* x^2+2x+1.

3. Substitute $x + 3$ for y in $y^4 - 2y^3 + y^2 - 6$.

Ans. $x^4 + 10x^3 + 37x^2 + 60x + 30$.

4. Substitute $s + r$ for x, in $x^2 + ax + b$, and arrange the result according to the descending powers of r.

Ans. $r^2 + (2s + a)r + s^2 + as + b$.

5. What will $a^4 + a^3b + a^2b^2 + ab^3 + b^4$ become, when $b = a$?

Ans. $5a^4$.

6. What will $x^3 + ax^2 + a^2x + a^3$ become, when $m + 1$ is put for x, and $m - 1$ for a? *Ans.* $4m(m^2 + 1)$.

7. What will $x^4 + y^4$ become, when $a + b$ is put for x, and $a - b$ for y? *Ans.* $2(a^4 + 6a^2b^2 + b^4)$.

8. What is the value of $(x + a + b + c)^5 + (x - a - b - c)^5$, when $a + b + c = s$? *Ans.* $2(x^5 + 10x^3s^2 + 5xs^4)$.

9. In $x^3 - 7x - 6$ substitute $y - 2$ for x. *Ans.* $y^3 - 6y^2 + 5y$.

10. In $x^5 - 2x^4 + 3x^3 - 7x^2 + 8x - 3$ substitute $y + 1$ for x.

Ans. $y^5 + 3y^4 + 5y^3$.

11. If $a - b = x$, $b - c = y$, and $c - a = z$, prove that $2(a-b)^2(b-c)^2 + 2(a-b)^2(c-a)^2 + 2(b-c)^2(c-a)^2 = x^4 + y^4 + z^4$.

THE GREATEST COMMON DIVISOR.

97. A ***Common Divisor*** of two or more quantities is a quantity which will exactly divide each of them.

98. The ***Greatest Common Divisor*** of two or more quantities is the greatest quantity that will exactly divide each of them; it is composed of all the common prime factors of the quantities.

The term, *greatest*, in this connection, is used in a qualified sense, and has reference to the *degree* of a quantity, or of its leading term, not to its algebraic or its arithmetical value. Thus, if $x - 3$ and $x^2 + 4x + 2$ are the prime factors common to two or more quantities, then according to the above definition, $(x^2 + 4x + 2)(x - 3) = x^3 + x^2 - 10x - 6$, is the greatest common divisor. But this product is not necessarily greater in value than one of the prime factors. For, if $x = 4$, then we have

$$x^2 + 4x + 2 = 34, \quad \text{and} \quad x^3 + x^2 - 10x - 6 = 34.$$

99. Several quantities are said to be *prime to each other* when they have no common factor.

CASE I.

100. When the given quantities can be factored by inspection.

It is evident (**81**, 2) that no factor of the greatest common divisor can have an exponent greater than the least with which it enters the given quantities. Hence the following obvious

RULE.—I. *Find by inspection, or otherwise, all the different prime factors that are common to the given quantities, and affect each with the least exponent which it has in any of the quantities.*

II. *Multiply together the factors thus obtained, and the product will be the greatest common divisor required.*

EXAMPLES FOR PRACTICE.

1. Find the greatest common divisor of $a^5 - 2a^3x^2 + ax^4$, and $a^4 - 2a^3x + a^2x^2$.

Factoring, we have

$$a^5 - 2a^3x^2 + ax^4 = a(a^4 - 2a^2x^2 + x^4) = a(a-x)^2(a+x)^2,$$
$$a^4 - 2a^3x + a^2x^2 = a^2(a^2 - 2ax + x^2) = a^2(a-x)^2.$$

The lowest powers of the common factors are a and $(a-x)^2$; and we have

$$a(a-x)^2 = a^3 - 2a^2x + ax^2$$

the greatest common divisor required.

2. Find the greatest common divisor of $2a^2bc^3$, $6ab^2c^5$, and $10a^3bc^2$. *Ans.* $2abc^2$.

3. Find the greatest common divisor of $5x^2y^2z^3$, $6x^3yz^2$, and $12x^2yz^3$. *Ans.* x^2yz^2.

4. Find the greatest common divisor of x^2-y^2 and $x^2-2xy+y^2$. *Ans.* $x-y$.

5. What is the greatest common divisor of $a^2m - b^2m$ and $2ac^2m - 2c^2bm$? *Ans.* $m(a-b)$.

6. What is the greatest common divisor of $a^2x^3 - 3a^2x^2 + a^2x$ and $3axz^2 - ax^2z^2 - az^2$? *Ans.* $a(x^2 - 3x + 1)$.

7. What is the greatest common divisor of $16x^2 - 1$, $x - 4x^2$, and $1 - 8x + 16x^2$? *Ans.* $4x - 1$.

Case II.

101. When the given quantities cannot be factored by inspection.

102. The greatest common divisor is found in this case by a process of decomposing the quantities by division. But in order to deduce a rule for the method, it will be necessary first to establish certain principles relating to exact division.

103. First, suppose A to be a quantity which is exactly divisible by another quantity, D, and let q represent the quotient. Then,

$$\frac{A}{D} = q.$$

If we now multiply the dividend by m, we shall have

$$\frac{Am}{D} = qm \text{ (84, I)},$$

in which qm is entire. Thus we have shown that if D divides A, it will also divide Am. Hence,

1. *If a quantity will exactly divide one of two quantities, it will divide their product.*

Again, let A and B represent any two quantities, and S their sum. Now suppose both A and B are exactly divisible by D, and let $\frac{A}{D} = q$, and $\frac{B}{D} = q'$. We shall have

$$A + B = S.$$

Dividing each term by D,

$$q + q' = \frac{S}{D},$$

in which $\frac{S}{D}$ must be entire, because its equal, $q + q'$, is entire. Hence,

2. *If a quantity will exactly divide each of two quantities, it will divide their sum.*

Finally, let d represent the difference of A and B, and suppose A and B to be divisible by D, q and q' being the quotients, as before, we shall have

$$A - B = d.$$

Dividing each term by D, $q - q' = \frac{d}{D}$,

in which $\frac{d}{D}$ is entire, because $q - q'$ is entire. Hence,

3. *If a quantity will exactly divide each of two quantities, it will divide their difference.*

104. We may now show, by the aid of these principles, what relation the greatest common divisor of two quantities bears to the parts of these quantities when decomposed by division.

Suppose two polynomials to be arranged according to the powers of the same letter, and let A represent the greater and B the less. Then let us divide the greater by the less, the last divisor by the last remainder, and so on, till nothing remains. If we represent the several quotients by q, q', q'', etc., and the remainders by R, R', R'', etc., the successive operations will appear as follows:

(1.)	(2.)	(3.)
$B\,)\,A\,(\,q$	$R\,)\,B\,(\,q'$	$R'\,)\,R\,(\,q''$
$\underline{Bq}$	$\underline{Rq'}$	$\underline{R'q''}$
R	R'	0

To investigate the mutual relations of A, B, R, and R', we observe that in division the product of the divisor and quotient, plus the remainder, if any, is equal to the dividend. Hence, from the operations above, we have the three following conditions:

$$\begin{aligned} R'q'' &= R, \\ Rq' + R' &= B, \\ Bq + R &= A. \end{aligned}$$

Now from the first equation it is evident that R' divides R without remainder; it will therefore divide Rq' (**103**, 1). And since R' divides both Rq' and itself, it must divide their sum, $Rq' + R'$, or B (**103**, 2); consequently, it will divide Bq (**103**, 1). Finally, since it divides both Bq and R, it must divide their sum, $Bq + R$, or A (**103**, 2). Hence,

I. *The last divisor, R', is a common divisor of R, B, and A, or of all the dividends.*

Again, the dividend minus the product of the divisor and quotient, is equal to the remainder. Therefore, from the first and second operations above, we have

$$\begin{aligned} A - Bq &= R, \\ B - Rq' &= R'. \end{aligned}$$

Now any expression which will divide B, will divide Bq (**103**, 1); hence, any expression which will divide both A and B, will also divide $A-Bq$, or R (**103**, 3). Whence it follows that the greatest common divisor of A and B will divide R, and is therefore a common divisor of B and R. For like reasons, referring to the second equation, the greatest common divisor of B and R will also divide R', and is therefore a common divisor of R and R'. But the *greatest* common divisor of R and R' is R' itself. Consequently, R' is the greatest common divisor of R and B, and also of B and A. Hence,

II. *The last divisor, R', is the greatest common divisor of the given quantities, and also of the dividend and divisor in each subsequent operation.*

1. What is the greatest common divisor of $12x^3-2x^2-7x-3$ and $3x^2-2x-1$?

FIRST OPERATION.

$$\begin{array}{r|l}
12x^3-2x^2-7x-3 & 3x^2-2x-1 \\ \cline{2-2}
12x^3-8x^2-4x & 4x+2 \\ \hline
6x^2-3x-3 & \\
6x^2-4x-2 & \\ \cline{1-1}
x-1 & \text{1st Rem.}
\end{array}$$

SECOND OPERATION.

$$\begin{array}{r|l}
3x^2-2x-1 & x-1 \\ \cline{2-2}
3x^2-3x & 3x+1 \\ \hline
x-1 & \\
x-1 & \\ \cline{1-1}
\end{array}$$

Ans. $x-1$.

The process here employed for finding the greatest common divisor of two polynomials, is subject to two modifications, which we will now investigate in their order.

1st. Suppressing Monomial Factors.

It is evident that any monomial factor *common* to the given polynomials, may be suppressed in both, and set aside as one factor of their greatest common divisor. We may then apply the process of division to the resulting polynomials, and obtain the remaining factor or factors of the greatest common divisor required.

Again, if either polynomial contains a factor which is *not common* to both, this factor can form no part of the greatest common

divisor required, and may therefore be suppressed. And since the greatest common divisor of the given polynomials is the same as that of the dividend and divisor in each operation following the first (II), it is evident that we may suppress the monomial factors in every *remainder* that occurs. And it should be observed, that if all the monomial factors of the given quantities have been previously suppressed, no monomial factor of any one of the remainders can belong to the greatest common divisor sought, or be common to any two successive remainders (II). This modification of the process will be illustrated by the example which follows:

2. What is the greatest common divisor of $12x^5 + 22x^3 + 6x$ and $6x^5 - 15x^3 - 36x$?

The first polynomial contains the monomial factor $2x$, and the second contains the monomial factor $3x$. We therefore suppress these factors, setting aside x, which is common, as one factor of the greatest common divisor sought. We then apply the process of division to the resulting polynomials, as follows:

FIRST OPERATION.

$$\begin{array}{r|l} 6x^4 + 11x^2 + 3 & 2x^4 - 5x^2 - 12 \\ \cline{2-2} \underline{6x^4 - 15x^2 - 36} & 3 \\ 26x^2 + 39 & \end{array}$$

Suppressing the factor 13 in this remainder, we have $2x^2 + 3$ for the next divisor.

SECOND OPERATION.

$$\begin{array}{r|l} 2x^4 - 5x^2 - 12 & 2x^2 + 3 \\ \cline{2-2} \underline{2x^4 + 3x^2 } & x^2 - 4 \\ - 8x^2 - 12 & \\ \underline{- 8x^2 - 12} & \end{array}$$

Taking the last divisor, and the common factor, x, which was set aside at the beginning, we have

$$(2x^2 + 3) \times x = 2x^3 + 3x, \; Ans.$$

2D. INTRODUCING MONOMIAL FACTORS.

It may happen at any stage of the process, that after suppressing every monomial factor of the divisor, its first term will not be exactly contained in the first term of the dividend. In such cases, the dividend may be multiplied by such a factor as will render its first term divisible by the first term of the divisor. No

factor thus introduced can be common to the dividend and divisor, since by hypothesis all the monomial factors of the divisor have previously been suppressed. Consequently, if the process of division be continued under this modification, the last divisor must be the greatest common divisor sought. This point will be illustrated by the following example :

3. What is the greatest common divisor of $2x^4 - 12x^3 + 17x^2 + 6x - 9$ and $4x^3 - 18x^2 + 19x - 3$?

We first multiply the greater polynomial by 2, to render its first term divisible by the first term of the other polynomial.

FIRST OPERATION.

$$
\begin{array}{r|l}
4x^4 - 24x^3 + 34x^2 + 12x - 18 & 4x^3 - 18x^2 + 19x - 3 \\
\underline{4x^4 - 18x^3 + 19x^2 - 3x \phantom{{}-18}} & \overline{x\,,\ -1} \\
-6x^3 + 15x^2 + 15x - 18 & \\
-2x^3 + 5x^2 + 5x - 6 & \\
-4x^3 + 10x^2 + 10x - 12 & \text{New prepared dividend.} \\
\underline{-4x^3 + 18x^2 - 19x + 3} & \\
-8x^2 + 29x - 15 &
\end{array}
$$

In the above operation, we suppress the factor 3 in the first remainder, and multiply the result by 2, to render the first term divisible by the first term of the divisor. We thus obtain $-4x^3 + 10x^2 + 10x - 12$ for the second dividend. As the two partial quotients, x and -1, have no connection, they are separated by a comma.

Multiplying the last divisor by 2 for a new dividend, we proceed as follows :

SECOND OPERATION.

$$
\begin{array}{r|l}
8x^3 - 36x^2 + 38x - 6 & -8x^2 + 29x - 15 \\
\underline{8x^3 - 29x^2 + 15x \phantom{{}-6}} & \overline{-x\,,\ +7} \\
-7x^2 + 23x - 6 & \\
-56x^2 + 184x - 48 & \text{New prepared dividend.} \\
\underline{-56x^2 + 203x - 105} & \\
-19x + 57 &
\end{array}
$$

Dividing this remainder by -19, we have $x - 3$ for the next divisor.

THIRD OPERATION.

$$
\begin{array}{r|l}
-8x^2+29x-15 & \underline{\;x-3\;} \\
\underline{-8x^2+24x} & -8x+5 \\
5x-15 & \\
\underline{5x-15} &
\end{array}
$$

Thus we find that the greatest common divisor is $x-3$. Had we suppressed $+19$ instead of -19, in the final remainder of the second operation, we should have obtained $-x+3$, or $3-x$ for the greatest common divisor. It should be remembered, however, that the term greatest, in this connection, has reference to exponents and coefficients, and not to the algebraic value (**98**). Consequently either $x-3$, or $3-x$ may be considered the greatest common divisor of the given polynomials. And it is immaterial what sign is given to any monomial factor which we may suppress or introduce at any stage of the work.

105. From these principles and illustrations we deduce the following general

RULE.—I. *Arrange the two polynomials with reference to the same letter; then suppress all the monomial factors of each, and if any factor suppressed is common to the two polynomials, set it aside as one factor of the common divisor sought.*

II. *Divide the greater of the resulting polynomials by the less, and continue the division till the first term of the remainder is of a lower degree than the first term of the divisor; observing to suppress the monomial factors in every remainder, and to introduce into any dividend, if necessary, such a factor as will render its first term exactly divisible by the first term of the divisor.*

III. *Take the final remainder in the first operation as a new divisor, and the former divisor as a new dividend, and proceed as before; and thus continue till the division is exact. The last divisor, multiplied by the common factor, if any, set aside at the beginning, will be the greatest common divisor required.*

IV. *If more than two polynomials are given, find the greatest common divisor of the first and second, and then the greatest common divisor of this result and the third polynomial, and so on. The last will be the greatest common divisor required.*

EXAMPLES FOR PRACTICE.

Find the greatest common divisor,

1. Of $x^4 - 2x^3 - 4x^2 + 11x - 6$ and $x^3 - 8x^2 + 17x - 10$. *Ans.* $x^2 - 3x + 2$.

2. Of $6x^3 + x^2 - 44x + 21$ and $6x^3 - 26x^2 + 46x - 42$. *Ans.* $3x - 7$.

3. Of $x^3 - 6ax^2 + 10a^2x - 3a^3$ and $3ax^2 - 14a^2x + 15a^3$. *Ans.* $x - 3a$.

4. Of $x^4 - 8x^3 + 14x^2 + 16x - 40$ and $x^3 - 8x^2 + 19x - 14$.

5. Of $a^3 + 5a^2 + 5a + 1$ and $a^3 + 1$. *Ans.* $a + 1$.

6. Of $2a^4 - 5a^3b - 3a^2b^2 + 7ab^3 + 3b^4$ and $4a^3 - 2a^2b - 4ab^2 - 3b^3$. *Ans.* $2a - 3b$.

7. Of $3x^3 - 4x^2y + 3xy^2 - 2y^3$ and $4x^2 - 7xy + 3y^2$. *Ans.* $x - y$.

8. Of $4x^5 - 2x^4 + 4x^3 - 27x^2 + 4x - 7$ and $2x^4 + 6x^3 - 19x^2 + 4x - 5$. *Ans.* $2x^3 - 4x^2 + x - 1$.

9. Of $a^5c - 4a^3cm + 3acm^2$ and $a^4c^2 - 6a^2c^2m + 5c^2m^2$. *Ans.* $c\,(a^2 - m)$.

10. Of $x^4 - 4x^3 - 16x^2 + 7x + 24$ and $2x^3 - 15x^2 + 9x + 40$. *Ans.* $x^2 - 5x - 8$.

11. Of $15x^6 + 71x^4 + 60x^2 - 56$ and $3x^6 - 17x^4 - 20x^2 + 84$. *Ans.* $3x^2 + 7$.

12. Of $3a^4 + 14a^2m^2 - 5m^4$, $6a^4 - 14a^2m^2 + 4m^4$, and $3a^4 - 22a^2m^2 + 7m^4$. *Ans.* $3a^2 - m^2$.

13. Of $2a^3x^3 - 2a^2bx^2y + ab^2xy^2 - b^3y^3$ and $a^2bx^2y - 2ab^2xy^2 + b^3y^3$. *Ans.* $ax - by$.

14. Of $9a^4 + 12a^3 + 10a^2 + 4a + 1$ and $3a^4 + 8a^3 + 14a^2 + 8a + 3$. *Ans.* $3a^2 + 2a + 1$.

LEAST COMMON MULTIPLE.

106. A ***Multiple*** of any quantity is another quantity exactly divisible by the given quantity.

It follows from this definition that if one quantity is a multiple of another, the multiple must be equal to the product of the other

quantity by some entire factor. Thus, if A is a multiple of B, then $A = Bm$, in which m is entire.

107. A ***Common Multiple*** of two or more quantities is one which is exactly divisible by each of them.

108. The ***Least Common Multiple*** of two or more quantities is the least quantity which is exactly divisible by each of them.

Case I.

109. When the quantities can be factored by inspection.

From the principles of exact division, we may make the following inferences:

1. A multiple of any quantity must contain all the factors of that quantity.
2. A common multiple of two or more quantities must contain all the factors of each of the quantities.
3. The least common multiple of two or more quantities must contain all the factors of each of the quantities, and no other factors.

Hence the following

Rule.—I. *Find by inspection all the different prime factors that enter into the given quantities, and affect each with an exponent equal to the greatest which it has in any of the quantities.*

II. *Multiply together the factors thus obtained, and the product will be the least common multiple required.*

EXAMPLES FOR PRACTICE.

1. What is the least common multiple of $a^2 + ab$, $a^2d - b^2d$, and $a^2c - 2abc + b^2c$?

Factoring, we have

$$a^2 + ab = a(a+b),$$
$$a^2d - b^2d = d(a-b)(a+b),$$
$$a^2c - 2abc + b^2c = c(a-b)^2.$$

The highest powers of the different prime factors are a, d, c, $(a-b)^2$, and $(a+b)$; and we have

$$acd(a-b)^2(a+b) = a^4cd - a^3bcd - a^2b^2cd + ab^3cd, \textit{ Ans.}$$

2. Find the least common multiple of $2a^4bc$, $5a^2c^2$, $10ab^3d$, and $15abcd$. *Ans.* $30a^4b^3c^2d$.

3. Find the least common multiple of $3x^2y$, $15xy^2$, $10xyz^3$, and $5x^3y^2z$. *Ans.* $30x^3y^2z^3$.

4. Find the least common multiple of $x^2 + xy$, $xy - y^2$, and $x^2 - y^2$. *Ans.* $x^3y - xy^3$.

5. Find the least common multiple of $x^4 - a^4$, $x^2 - a^2$, $x^2 + a^2$, and $x^4 - 2a^2x^2 + a^4$. *Ans.* $x^6 - a^2x^4 - a^4x^2 + a^6$.

6. Find the least common multiple of $x^3 - x$, $x^3 - 1$, and $x^3 + 1$.

7. Find the least common multiple of $x^4 + 2x^2 + 1$, $x^4 - 2x^2 + 1$, $x^2 + 2x + 1$, $x^2 - 2x + 1$, $x + 1$, and $x - 1$. *Ans.* $x^8 + 2x^4 + 1$.

8. What is the least common multiple of $4x^3 + 2x$, $6x^2 - 4x$, and $6x^2 + 4x$? *Ans.* $36x^5 + 2x^3 - 8x$.

9. What is the least common multiple of $x^2 - 4a^2$, $(x + 2a)^3$, and $(x - 2a)^3$?

10. What is the least common multiple of $a^4 - b^4$, $a^3 - b^3$, $a^2 - b^2$, and $a - b$? *Ans.* $a^6 + a^5b + a^4b^2 - a^2b^4 - ab^5 - b^6$.

CASE II.

101. When the quantities cannot be factored by inspection.

The rule for this case may be deduced as follows:

1. If two polynomials are prime to each other, their product must be their least common multiple.

2. If two polynomials have a common divisor, their product must contain the second power of this common divisor; their least common multiple may therefore be obtained by suppressing the first power of the greatest common divisor in the product, or in one of the given quantities before multiplication.

3. If we find the least common multiple of two polynomials, and then the least common multiple of this result and a third polynomial, and so on, the last result will evidently contain all the factors of the given polynomials, and no other factors. It will, therefore, be the least common multiple of the polynomials (**109**, 3).

Hence the following

Rule.—I. When only two polynomials are given:

Find the greatest common divisor of the given polynomials; suppress this divisor in one of the polynomials, and multiply the result by the other polynomial.

II. When three or more polynomials are given:

Find the least common multiple of any two of the polynomials; then find the least common multiple of this result and a third polynomial; and so on, till all the polynomials have been used. The last result will be the least common multiple required.

Note.—It will generally be found preferable to commence with the greatest and next greatest of the given quantities.

EXAMPLES FOR PRACTICE.

Find the least common multiple

1. Of $x^3 + x^2 - 4x + 6$ and $x^3 - 5x^2 + 8x - 6$.
   Ans. $x^4 - 2x^3 - 7x^2 + 18x - 18$.

2. Of $x^3 - 2x^2 - 19x + 20$ and $x^2 - 12x + 35$.
   Ans. $x^4 - 9x^3 - 5x^2 + 153x - 140$.

3. Of $6a^2m^4 - am^2 - 1$ and $2a^2m^4 + 3am^2 - 2$.
   Ans. $6a^3m^6 + 11a^2m^4 - 3am^2 - 2$.

4. Of $2x^3 - 5x^2 - x + 1$ and $x^3 - 5x^2 + 7x - 2$.
   Ans. $2x^4 - 9x^3 + 9x^2 + 3x - 2$.

5. Of $3x^3 + 6x^2 - 5x - 10$ and $6x^4 - 4x^2 - 10$.
   Ans. $6x^5 + 12x^4 - 4x^3 - 8x^2 - 10x - 20$.

6. Of $x^2 + 7x + 10$, $x^2 - 2x - 8$, and $x^2 + x - 20$.
   Ans. $x^3 + 3x^2 - 18x - 40$.

7. Of $a^2 - 3ab + 2b^2$, $a^2 - ab - 2b^2$, and $a^2 - b^2$.
   Ans. $a^3 - 2a^2b - ab^2 + 2b^3$.

8. Of $2x^2 - 7xy + 3y^2$, $2x^2 - 5xy + 2y^2$, and $x^2 - 5xy + 6y^2$.
   Ans. $2x^3 - 11x^2y + 17xy^2 - 6y^3$.

FRACTIONS.

DEFINITIONS AND NOTATION.

111. We have seen (**12**) that division may be indicated by writing the dividend and divisor on opposite sides of a horizontal line. The term *Fraction*, in Algebra, relates to this mode or form of indicating division. Hence,

112. A ***Fraction*** is a quotient expressed by writing the dividend above a horizontal line, and the divisor below. Thus $\frac{a}{b}$ is a fraction, and is read, *a* divided by *b*.

113. The ***Denominator*** of the fraction is the quantity below the line, or the divisor.

114. The ***Numerator*** is the quantity above the line, or the dividend.

115. Any fraction may be decomposed as follows:

$$\frac{a}{b} = \frac{1 \times a}{b} = \left(\frac{1}{b}\right) \times a.$$

Hence,

1. The value of a fraction is equal to the reciprocal of the denominator multiplied by the numerator.

2. In any fraction, the reciprocal of the denominator may be regarded as a *fractional unit;* and the numerator shows how many times this unit is taken in the fraction. Hence,

3. A fraction is a *fractional unit* or a *collection of fractional units*, the value of each depending upon the denominator.

116. An ***Entire Quantity*** is an algebraic expression which has no fractional part; as $x^2 - 3xy$.

117. A ***Mixed Quantity*** is one which has both entire and fractional parts; as $a^2 + \frac{x}{b}$.

GENERAL PRINCIPLES OF FRACTIONS.

118. Since a fraction is a form of expressing division, it is evident that all the operations in fractions must be based upon the general relations subsisting between the dividend, divisor, and quotient. These principles relate, first, to change of value; second, to change of sign.

1st. Change of Value.

119. By modifying the language of **84**, we may express the mutual relations of the numerator and denominator of a fraction, as follows:

I. *Multiplying the numerator multiplies the fraction, and dividing the numerator divides the fraction.*

II. *Multiplying the denominator divides the fraction, and dividing the denominator multiplies the fraction.*

III. *Multiplying or dividing both numerator and denominator by the same quantity does not alter the value of the fraction.*

2d. Change of Sign.

120. The ***Apparent Sign*** of a fraction is the sign written before the dividing line, to indicate whether the fraction is to be added or subtracted. Thus, in

$$y + \frac{a^2 - ax^2}{4a - 2x}$$

the apparent sign of the fraction is plus, and indicates that the fraction is to be added.

121. The ***Real Sign*** of a fraction is the sign of its numerical value, when reduced to a monomial, and shows whether the fraction is essentially a positive or a negative quantity. Thus, in the fraction just given, let $a = 2$ and $x = 3$. Then

$$\frac{a^2 - ax^2}{4a - 2x} = \frac{4 - 18}{8 - 6} = \frac{-14}{2} = -7.$$

The *real* sign of this fraction therefore is minus, though its *apparent* sign is plus.

122. Each term in the numerator and denominator of a fraction has its own particular sign, distinct from the real or apparent

sign of the fraction. Now the essential sign of any entire quantity is changed, by changing the signs of all its terms. Hence,

I. *Changing all the signs of either numerator or denominator changes the real sign of the fraction* (**85**, I).

II. *Changing all the signs of both numerator and denominator does not alter the real sign of the fraction* (**85**, II).

III. *Changing the apparent sign of the fraction changes the real sign.*

REDUCTION.

123. The ***Reduction*** of a fraction is the operation of changing its form without altering its value.

CASE I.

124. To reduce a fraction to its lowest terms.

A fraction is in its lowest terms, when the numerator and denominator are prime to each other. And since it does not alter the value of a fraction to suppress the same factor in both numerator and denominator (**119**, III), we have the following

RULE.—I. *Resolve the numerator and denominator into their prime factors, and cancel all those factors which are common.* Or,

II. *Divide both numerator and denominator by their greatest common divisor.*

EXAMPLES FOR PRACTICE.

1. Reduce $\frac{a^4-1}{a^5+a^3}$ to its lowest terms.

$$\frac{a^4-1}{a^5+a^3}=\frac{(a^2-1)(a^2+1)}{a^3(a^2+1)}=\frac{a^2-1}{a^3}, \textit{ Ans.}$$

2. Reduce $\frac{3a^2-2a-1}{4a^3-2a^2-3a+1}$ to its lowest terms.

The greatest common divisor of the numerator and denominator, as found by **105**, is $a-1$; hence,

$$(3a^2-2a-1)\div(a-1)=3a+1,$$
$$(4a^3-2a^2-3a+1)\div(a-1)=4a^2+2a-1;$$

and we have for the reduced fraction,

$$\frac{3a+1}{4a^2+2a-1}, \textit{ Ans.}$$

Reduce each of the following fractions to its lowest terms:

3. $\dfrac{7x^3yz}{21xy^2z}$. *Ans.* $\dfrac{x^2}{3y^2}$.

4. $\dfrac{x^2-1}{xy+y}$. *Ans.* $\dfrac{x-1}{y}$.

5. $\dfrac{a^3-ab^2}{a^2+2ab+b^2}$. *Ans.* $\dfrac{a^2-ab}{a+b}$.

6. $\dfrac{x^5-b^2x^3}{x^4-b^4}$. *Ans.* $\dfrac{x^3}{x^2+b^2}$.

7. $\dfrac{2x^3-16x-6}{2x^3-24x-9}$. *Ans.* $\dfrac{2}{3}$.

8. $\dfrac{2x^3-7x^2+14x-12}{4x^3-4x^2-13x+15}$. *Ans.* $\dfrac{x^2-2x+4}{2x^2+x-5}$.

9. $\dfrac{a^2c+2abc+b^2c}{a^3+3a^2b+3ab^2+b^3}$. *Ans.* $\dfrac{c}{a+b}$.

10. $\dfrac{a^3-3a^2x+3ax^2-x^3}{a^2-x^2}$. *Ans.* $\dfrac{a^2-2ax+x^2}{a+x}$.

11. $\dfrac{6a^2+7ax-3x^2}{6a^2+11ax+3x^2}$. *Ans.* $\dfrac{3a-x}{3a+x}$.

12. $\dfrac{x^5-x^4-x+1}{x^4-x^3-x^2+x}$. *Ans.* $\dfrac{x^2+1}{x}$.

13. $\dfrac{(x+y)^5-x^5-y^5}{(x+y)^3-x^3-y^3}$. *Ans.* $\dfrac{5(x^2+xy+y^2)}{3}$.

14. $\dfrac{(3x^2-1)(2x^2-1)-x^2(5x^2-7)}{(3x^2-1)^2+(x^3-3x)^2}$. *Ans.* $\dfrac{1}{x^2+1}$.

CASE II.

125. To reduce a fraction to an entire or mixed quantity.

The division indicated by a fraction may be at least partially performed, when there is any term in the numerator whose *literal part* is exactly divisible by some term in the denominator. Hence,

RULE I.—*Divide the numerator by the denominator as far as possible; the quotient will be the entire quantity.*

II. *Write the remainder over the denominator, annex the fraction thus formed to the entire part, with its proper sign, and the whole result will be the mixed quantity.*

EXAMPLES FOR PRACTICE.

Reduce the following fractions to entire or mixed quantities:

1. $\dfrac{ab + x}{b}$. *Ans.* $a + \dfrac{x}{b}$.

2. $\dfrac{a^2 + bx}{a}$. *Ans.* $a + \dfrac{bx}{a}$.

3. $\dfrac{5ay + ab + y}{y}$. *Ans.* $5a + 1 + \dfrac{ab}{y}$.

4. $\dfrac{2x^3 - 2y^3}{x - y}$. *Ans.* $2(x^2 + xy + y^2)$.

5. $\dfrac{3x^2 - 12ax - 9x + y}{3x}$. *Ans.* $x - 4a - 3 + \dfrac{y}{3x}$.

6. $\dfrac{24x^2 - 18x - 6}{8x}$. *Ans.* $3x - 2 - \dfrac{x + 3}{4x}$.

7. $\dfrac{3x^3 - 7x^2 + 7x + 30}{x^2 - 4x + 8}$. *Ans.* $3x + 5 + \dfrac{3x - 10}{x^2 - 4x + 8}$.

8. $\dfrac{56x^2 + 126x - 140}{7x + 21}$. *Ans.* $8x - 6 - \dfrac{2}{x + 3}$.

9. $\dfrac{x^5 + y^5}{x - y}$. *Ans.* $x^4 + x^3y + x^2y^2 + xy^3 + y^4 + \dfrac{2y^5}{x - y}$.

10. $\dfrac{x^7 - y^7}{x^3 - y^3}$. *Ans.* $x^4 + xy^3 + \dfrac{y^6}{x^2 + xy + y^2}$.

11. $\dfrac{x^6 - 6x^4 + 10x^2 - 3}{x^2 - 1}$. *Ans.* $x^4 - 5x^2 + 5 + \dfrac{2}{x^2 - 1}$.

CASE III.

126. To reduce a mixed quantity to the form of a fraction.

This case is the converse of the last, and may be explained by it.

Hence the following

RULE.—*Multiply the entire part by the denominator of the fraction; add the numerator if the sign of the fraction be plus, but subtract it if the sign be minus, and write the result over the denominator.*

EXAMPLES FOR PRACTICE.

Reduce the following mixed quantities to fractions:

1. $1 + a + \dfrac{a^2}{b}$. *Ans.* $\dfrac{b + ab + a^2}{b}$.

2. $2b - \dfrac{3x - a}{c}$. *Ans.* $\dfrac{2bc - 3x + a}{c}$.

3. $5a + \dfrac{ab + x}{b}$. *Ans.* $\dfrac{6ab + x}{b}$.

4. $12 + \dfrac{3a + b}{b}$. *Ans.* $\dfrac{13b + 3a}{b}$.

5. $5x - \dfrac{2x - 5}{3}$. *Ans.* $\dfrac{13x + 5}{3}$.

6. $3a - 9 - \dfrac{3a^2 - 30}{a + 3}$. *Ans.* $\dfrac{3}{a + 3}$.

7. $x + y + \dfrac{y^2}{x - y}$. *Ans.* $\dfrac{x^2}{x - y}$.

8. $x + 1 - \dfrac{x^3 - 4x^2 + 8}{(x - 2)^2}$. *Ans.* $\dfrac{x + 2}{x - 2}$.

9. $a^2 + ab + b^2 - \frac{a^3 + b^3}{a - b}$. *Ans.* $-\frac{2b^3}{a - b}$.

10. $1 + 2y + 2y^2 + 2y^3 + \frac{2y^4 + 2y^5}{1 - y^2}$. *Ans.* $\frac{1 + y}{1 - y}$.

11. $(x - 1)^3 - \frac{(x - 1)^3}{x}$. *Ans.* $\frac{(x - 1)^4}{x}$.

12. $x^2 + 5xy + y^2 + \frac{21x^2y^2}{x^2 - 5xy + y^2}$. *Ans.* $\frac{(x^2 - y^2)^2}{x^2 - 5xy + y^2}$.

CASE IV.

127. To transfer a factor from the denominator to the numerator, or the reverse.

Let us take any fraction, as $\frac{ax^m}{by^n}$, and multiply both numerator and denominator by y^{-n}, observing that any factor having zero for its exponent is equal to unity (**88**, 1), and may therefore be omitted. We shall have

$$\frac{ax^m}{by^n} = \frac{ax^my^{-n}}{by^{n-n}} = \frac{ax^my^{-n}}{by^0} = \frac{ax^my^{-n}}{b}.$$

If we multiply both numerator and denominator of the same fraction by x^{-m}, we shall have

$$\frac{ax^m}{by^n} = \frac{ax^{m-m}}{by^nx^{-m}} = \frac{ax^0}{by^nx^{-m}} = \frac{a}{by^nx^{-m}}.$$

In like manner we may transfer any factor having a negative exponent. For example, let us take the fraction, $\frac{ax^{-s}}{b}$, and multiply both numerator and denominator by x^s; we shall have

$$\frac{ax^{-s}}{b} = \frac{ax^{-s+s}}{bx^s} = \frac{ax^0}{bx^s} = \frac{a}{bx^s}.$$

By the same principle, also, any fraction may be reduced to the form of an entire quantity; thus,

$$\frac{x^m}{y^n} = \frac{x^my^{-n}}{y^{n-n}} = \frac{x^my^{-n}}{y^0} = \frac{x^my^{-n}}{1} = x^my^{-n}.$$

In all operations of this kind, the intermediate steps may be omitted, and the results obtained by the following

RULE.—I. To transfer any factor from the denominator to the numerator, or the reverse :—*Change the sign of its exponent.*

II. To reduce any fraction to the form of an entire quantity :—*Transfer all the factors of the denominator to the numerator, observing to change the signs of the exponents of the factors transferred.*

EXAMPLES FOR PRACTICE.

In each of the following fractions, transfer the unknown factors, or factors containing unknown quantities, to the numerator.

1. $\frac{ax}{c^2y^2}$. *Ans.* $\frac{axy^{-2}}{c^2}$.

2. $\frac{3a^2}{5mx^3}$. *Ans.* $\frac{3a^2x^{-3}}{5m}$.

3. $\frac{1}{axyz}$. *Ans.* $\frac{x^{-1}y^{-1}z^{-1}}{a}$.

4. $\frac{c}{a(x-y)}$. *Ans.* $\frac{c(x-y)^{-1}}{a}$.

5. $\frac{2a^2x^2y^3}{5a^2xy^7}$. *Ans.* $\frac{2a^2xy^{-4}}{5a^2}$.

6. $\frac{4x^2z}{3ax^{-3}}$. *Ans.* $\frac{4x^5z}{3a}$.

7. $\frac{3b^2(a-x)}{5cm(a-x)^{-2}}$. *Ans.* $\frac{3b^2(a-x)^3}{5cm}$.

8. $\frac{3c^2(1-x)^{-2}(x-y)}{4m(x-y)^2(1-x)}$. *Ans.* $\frac{3c^2(1-x)^{-3}(x-y)^{-1}}{4m}$.

In each of the following fractions, transfer the known factors to the denominator, and the unknown factors to the numerator:

9. $\frac{a^2bc^2x^3}{5xy^2z}$. *Ans.* $\frac{x^2y^{-2}z^{-1}}{5a^{-2}b^{-1}c^{-2}}$.

10. $\frac{(a-b)(x-a)^2}{(x-a)^{-1}(a-b)^3}$. *Ans.* $\frac{(x-a)^3}{(a-b)^2}$.

11. $\frac{5a^2bx^3y^2}{cb^{-2}x^5y^{-4}}$. *Ans.* $\frac{x^{-2}y^6}{5^{-1}a^{-2}b^{-3}c}$.

In the following fractions, transfer the factors having negative exponents.

12. $\frac{3a^{-2}xz}{5mz^{-3}}$. *Ans.* $\frac{3xz^4}{5a^2m}$.

13. $\frac{5x(x^2-1)}{3ax^{-3}(x^2-1)^{-1}}$. *Ans.* $\frac{5x^4(x^2-1)^2}{3a}$.

14. $\frac{5ab^{-12}cd^{-3}}{12a^{-4}b^{-7}c^{-3}}$. *Ans.* $\frac{5a^5c^4}{12b^5d^3}$.

Reduce each of the following fractions to the form of an entire quantity:

15. $\frac{5a^2b}{x^3z}$. *Ans.* $5a^2bx^{-3}z^{-1}$.

16. $\frac{7xy^2}{a^2m^3}$. *Ans.* $7a^{-2}m^{-3}xy^2$.

17. $\frac{a^2b}{4x^3y}$. *Ans.* $4^{-1}a^2bx^{-3}y^{-1}$.

18. $\frac{4ab^2}{(a-x)^2}$. *Ans.* $4ab^2(a-x)^{-2}$.

CASE V.

128. To reduce two or more fractions to a common denominator.

We have seen (**124**) that a fraction may be reduced to lower terms by division. Conversely, a fraction must be reduced to higher terms by multiplication, and each of the higher denominators it may have must be some multiple of its lowest denominator. Hence,

1. A common denominator to which two or more fractions may be reduced, must be a common multiple of their lowest denominators; and

2. The *least* common denominator of two or more fractions, must be the *least* common multiple of their denominators.

1. Reduce $\frac{c}{a^2b}$ and $\frac{d}{ab^2}$, to their least common denominator.

We find by inspection that the least common multiple of the given denominators is a^2b^3. And

$$a^2b^3 \div a^2b = b^2,$$
$$a^2b^3 \div ab^3 = a.$$

If, therefore, we multiply both numerator and denominator of the first fraction by b^2, and of the second by a, we shall reduce the two fractions to their least common denominator, a^2b^3. Thus,

$c \times b^2 = b^2c$, new numerator of first fraction;
$d \times a = ad$, new numerator of second fraction.

Hence $\frac{c}{a^2b}, \frac{d}{ab^3} = \frac{b^2c}{a^2b^3}, \frac{ad}{a^2b^3}$ *Ans.*

From these principles and illustrations we deduce the following

RULE.—I. *Find the least common multiple of all the denominators, for the least common denominator.*

II. *Divide this common denominator by each of the given denominators, and multiply each numerator by the corresponding quotient. The products will be the new numerators.*

NOTE.—Mixed numbers should first be reduced to fractions, and all fractions to their lowest terms.

EXAMPLES FOR PRACTICE.

In each of the following examples, reduce the fractions and mixed quantities to their least common denominator:

1. $\frac{2a}{x}$ and $\frac{3b}{2c}$. *Ans.* $\frac{4ac}{2cx}, \frac{3bx}{2cx}$.

2. $\frac{2a}{b}$ and $\frac{3a + 2b}{2c}$. *Ans.* $\frac{4ac}{2bc}, \frac{3ab + 2b^2}{2bc}$.

3. $\frac{5a}{3x}, \frac{3b}{2c}$, and $4d$. *Ans.* $\frac{10ac}{6cx}, \frac{9bx}{6cx}, \frac{24cdx}{6cx}$.

4. $\frac{a}{b}, \frac{x + 1}{c}$, and $\frac{y}{x + a}$.

Ans. $\frac{acx + a^2c}{bcx + abc}, \frac{(bx + b)(x + a)}{bcx + abc}, \frac{bcy}{bcx + abc}$.

5. $a^3 + \frac{a}{y}$ and $\frac{c}{ay - 1}$. *Ans.* $\frac{a^3y^2 - a}{ay^2 - y}, \frac{cy}{ay^2 - y}$.

6. $\frac{x}{a+b}$, $\frac{y}{a-b}$, and $\frac{z}{a^2-b^2}$.

Ans. $\frac{(a-b)x}{a^2-b^2}$, $\frac{(a+b)y}{a^2-b^2}$, $\frac{z}{a^2-b^2}$.

7. $\frac{a}{x-1}$, $\frac{b}{x^2-1}$, and $\frac{c}{x^4-1}$.

Ans. $\frac{a(x^3+x^2+x+1)}{x^4-1}$, $\frac{b(x^2+1)}{x^4-1}$, $\frac{c}{x^4-1}$.

8. $\frac{x^2+x+1}{x^3-6x^2+6x-5}$ and $\frac{x^2-x+1}{x^3-4x^2-4x-5}$.

Ans. $\frac{(x^2+x+1)^2}{x^5-5x^4+x^3-5x^2+x-5}$, $\frac{(x^2-x+1)^2}{x^5-5x^4+x^3-5x^2+x-5}$.

ADDITION.

129. We have seen **(115)** that a fraction is equal to the reciprocal of its denominator multiplied by the numerator. Hence, if two or more fractions have a common denominator, they will have a common fractional unit, which may be made the unit of addition. Thus,

$$\frac{a}{c}+\frac{b}{c}=\frac{1}{c}\times a+\frac{1}{c}\times b=\frac{1}{c}\times(a+b)=\frac{a+b}{c}.$$

Or thus,

$$\frac{a}{c}+\frac{b}{c}=ac^{-1}+bc^{-1}=(a+b)c^{-1}=\frac{a+b}{c}.$$

The intermediate steps may be omitted; hence the following

RULE.—I. *Reduce the fractions to their least common denominator.*

II. *Add the numerators, and write the result over the common denominator.*

NOTES.—1. If there are mixed quantities, we may add the entire and fractional parts separately.

2. Any fractional result should be reduced to its lowest terms.

EXAMPLES FOR PRACTICE.

1. Add $\frac{3x}{5}$, $\frac{2x}{7}$, and $\frac{x}{3}$. *Ans.* $\frac{63x + 30x + 35x}{105} = \frac{128x}{105}$.

2. Add $\frac{a}{b}$ and $\frac{a + b}{c}$. *Ans.* $\frac{ac + ab + b^2}{bc}$.

3. Add $\frac{a^2}{3}$ and $\frac{a^2 + x^2}{a + x}$. *Ans.* $\frac{a^3 + a^2x + 3a^2 + 3x^2}{3(a + x)}$.

4. Add $\frac{a + b}{a - b}$ and $\frac{a - b}{a + b}$. *Ans.* $\frac{2a^2 + 2b^2}{a^2 - b^2}$.

5. Add $2a + \frac{a + 3}{5}$ and $4a + \frac{2a - 5}{4}$. *Ans.* $6a + \frac{14a - 13}{20}$.

6. Add $5x + \frac{x - 2}{3}$ and $4x + \frac{2x - 3}{5x}$. *Ans.* $9x + \frac{5x^2 - 4x - 9}{15x}$.

7. Add $\frac{a}{a + c}$, $\frac{2c}{a - c}$, and $\frac{c}{a + c}$. *Ans.* $\frac{a + c}{a - c}$.

8. Add $\frac{x^2y - 3y^2}{5x^2}$, $\frac{3x^4 + 3y^4}{5x^2y^2}$, and $\frac{xy^2 - 6x^3}{10y^2}$. *Ans.* $\frac{x + 2y}{10}$.

9. Add $\frac{a + b}{(b - c)(c - a)}$, $\frac{b + c}{(c - a)(a - b)}$, and $\frac{c + a}{(a - b)(b - c)}$.
Ans. 0.

10. Add $\frac{a^2 - b}{(a - b)(a - 1)}$, $\frac{b^2 + a}{(b + 1)(b - a)}$, and $\frac{1 + ab}{(1 - a)(1 + b)}$.
Ans. 0.

11. Add $\frac{bc}{(a - b)(a - c)}$, $\frac{ac}{(b - c)(b - a)}$, and $\frac{ab}{(c - a)(c - b)}$.
Ans. 1.

12. Add $\frac{x - 3}{x^2 - 3x + 2}$, $\frac{x - 2}{x^2 - 4x + 3}$, and $\frac{x - 1}{x^2 - 5x + 6}$.
Ans. $\frac{3x^2 - 12x + 14}{x^3 - 6x^2 + 11x - 6}$.

13. Add $\frac{x}{x + 1}$, $\frac{x^2}{x^2 + 3x + 2}$, and $\frac{x^3 - 2x^2 - 3x}{x^3 + 6x^2 + 11x + 6}$.
Ans. $\frac{3x(x + 1)}{x^2 + 5x + 6}$.

SUBTRACTION.

130. If two fractions have a common denominator, they will have the same fractional unit; and the one may be subtracted from the other, by taking the difference of the numerators. Thus,

$$\frac{a}{c}-\frac{b}{c}=\frac{1}{c}\times a-\frac{1}{c}\times b=\frac{1}{c}\times(a-b)=\frac{a-b}{c}.$$

Or thus,

$$\frac{a}{c}-\frac{b}{c}=ac^{-1}-bc^{-1}=(a-b)c^{-1}=\frac{a-b}{c}.$$

Hence the following

RULE.—I. *Reduce the fractions to their least common denominator.*

II. *Subtract the numerator of the subtrahend from the numerator of the minuend, and write the result over the common denominator.*

EXAMPLES FOR PRACTICE.

1. From $\frac{3x}{7}$ subtract $\frac{2x}{9}$. *Ans.* $\frac{13x}{63}$.

2. From $\frac{7x}{2}$ subtract $\frac{2x-1}{3}$. *Ans.* $\frac{17x+2}{6}$.

3. From $\frac{1}{x-y}$ subtract $\frac{1}{x+y}$. *Ans.* $\frac{2y}{x^2-y^2}$.

4. From $3a+\frac{11a-10}{15}$ take $2a+\frac{3a-5}{7}$.

5. From $\frac{a+b}{a-b}$ take $\frac{a-b}{a+b}$. *Ans.* $\frac{4ab}{a^2-b^2}$.

6. From $x+\frac{x-y}{x^2+xy}$ take $\frac{x+y}{x^2-xy}$. *Ans.* $x-\frac{4y}{x^2-y^2}$.

7. From $3x+\frac{x}{b}$ take $x-\frac{x-a}{c}$.

8. From $\frac{x^2+x-5}{2x^2-11x+12}$ take $\frac{x^2+x-1}{2x^2+5x-12}$. *Ans.* $\frac{4x+8}{x^2-16}$.

9. From $\dfrac{3a+b}{a^2+3ab+2b^2}$ take $\dfrac{a+7b}{a^2+5ab+6b^2}$.

Ans. $\dfrac{2(a-b)}{a^2+4ab+3b^2}$.

10. From $\dfrac{4a-3b}{7ab(a-b)-2(a^3-b^3)}$ take $\dfrac{8a-b}{3ab(a+b)-2(a^3+b^3)}$.

Ans. $\dfrac{2}{a^2-b^2}$.

MULTIPLICATION.

131. Any fraction may be multiplied by an entire quantity in two ways:

1st. By multiplying its numerator; or

2d. By dividing its denominator (**119,** I and II).

132. A general rule for the multiplication of fractions is furnished by the following example:

1. Multiply $\frac{a}{b}$ by $\frac{c}{d}$.

OPERATION.

$$\frac{a}{b} \times \frac{c}{d} = ab^{-1} \times cd^{-1} = acb^{-1}d^{-1} = \frac{ac}{bd}.$$

By observing the result, we find that the new numerator is the product of the given numerators, and the new denominator is the product of the given denominators. Hence the following

RULE.—I. *Reduce entire and mixed quantities to fractional forms.*

II. *Multiply the numerators together for a new numerator, and the denominators for a new denominator, canceling all factors common to the numerator and denominator of the indicated product.*

EXAMPLES FOR PRACTICE.

1. Multiply $\frac{a}{b}$ by $\frac{b}{x}$. Ans. $\frac{a}{x}$.

2. Multiply $\dfrac{a+x}{30}$ by $\dfrac{5a}{3(a+x)}$. Ans. $\dfrac{a}{18}$.

3. Multiply $\frac{2x + 3y}{2a}$ by $\frac{2a}{5x}$. *Ans.* $\frac{2x + 3y}{5x}$.

4. Multiply $\frac{a^2 - x^2}{2y}$ by $\frac{2a}{a + x}$. *Ans.* $\frac{(a - x)a}{y}$.

5. Multiply $\frac{4y^2}{5y - 10}$ by $\frac{15y - 30}{2y}$. *Ans.* $6y$.

6. Multiply $\frac{a^4 - b^4}{a + b}$ by $\frac{a^2}{ab - b^2}$. *Ans.* $\frac{a^2(a^2 + b^2)}{b}$.

7. Multiply $\frac{a^2x - x^3}{a}$ by $\frac{6a}{2ax - 2x^2}$. *Ans.* $3(a + x)$.

8. Multiply $a + \frac{x}{b}$ by $a - \frac{y}{b}$. *Ans.* $\frac{a^2b^2 + abx - aby - xy}{b^2}$.

9. Multiply $\frac{3x^2 - 5x}{14}$ by $\frac{7a}{2x^3 - 3x}$. *Ans.* $\frac{3ax - 5a}{4x^2 - 6}$.

10. Multiply together $\frac{x^2 - y^2}{x}$, $\frac{x}{x + y}$, and $\frac{a}{x - y}$. *Ans.* a.

11. Multiply $\frac{4a^2 - 16b^2}{a - 2b}$ by $\frac{5b}{8a^2 + 32ab + 32b^2}$. *Ans.* $\frac{5b}{2a + 4b}$.

12. Multiply together $\frac{x + 1}{2a}$, $\frac{x - 1}{a + b}$, and $3a$. *Ans.* $\frac{3(x^2 - 1)}{2(a + b)}$.

13. Multiply together $\frac{a^2 - x^2}{a + b}$, $\frac{a^2 - b^2}{ax + x^2}$, and $a + \frac{ax}{a - x}$.

Ans. $\frac{a^2(a - b)}{x}$.

14. Multiply together $\frac{a^4 - x^4}{a^2 - b^2}$, $\frac{a + b}{a^2 + x^2}$, and $\frac{a - b}{a - x}$. *Ans.* $a + x$.

15. Multiply together $\frac{x^2 - b^2}{bc}$, $\frac{x^2 + b^2}{b + c}$, and $\frac{bc}{x - b}$.

Ans. $\frac{(x + b)(x^2 + b^2)}{b + c}$.

16. Multiply together $\dfrac{c(a-c)}{a^2+2ac+c^2}$, $\dfrac{c(a+c)}{a^2-2ac+c^2}$, and $\dfrac{a^2-c^2}{ac^2x}$.

Ans. $\dfrac{1}{ax}$.

17. Multiply $\dfrac{(a+b-c)(a-b+c)}{a-b-c}$ by $\dfrac{c+b-a}{(c-b-a)(b-c-a)}$.

Ans. -1.

DIVISION.

133. Any fraction may be divided by an entire quantity in two ways:

1st. By dividing its numerator; or

2d. By multiplying its denominator (**119,** I and II).

We may, however, derive a general rule for the division of fractions, from the following example:

1. Divide $\dfrac{a}{b}$ by $\dfrac{c}{d}$.

OPERATION.

$$\frac{a}{b} \div \frac{c}{d} = ab^{-1} \div cd^{-1} = \frac{ab^{-1}}{cd^{-1}} = \frac{ad}{bc}.$$

By inspecting this result, we find that the new numerator may be obtained, by multiplying the numerator of the dividend by the denominator of the divisor; and the new denominator may be obtained, by multiplying the denominator of the dividend by the numerator of the divisor. Hence the

RULE.—I. *Reduce entire and mixed quantities to fractional forms.*

II. *Invert the terms of the divisor, and proceed as in multiplication.*

EXAMPLES FOR PRACTICE.

1. Divide $\dfrac{5x}{a}$ by $\dfrac{b}{c}$. *Ans.* $\dfrac{5x}{a} \times \dfrac{c}{b} = \dfrac{5cx}{ab}$.

2. Divide $\dfrac{a+b}{c}$ by $\dfrac{c}{a+b}$. *Ans.* $\dfrac{(a+b)^2}{c^2}$.

3. Divide $\frac{15ab}{a-x}$ by $\frac{10ac}{a^2-x^2}$. *Ans.* $\frac{3b(a+x)}{2c}$.

4. Divide $\frac{2x^2-7}{x+a}$ by $\frac{a^2}{x^2+2ax+a^2}$. *Ans.* $\frac{(2x^2-7)(x+a)}{a^2}$.

5. Divide $\frac{x^4-b^4}{x^2-2bx+b^2}$ by $\frac{x+b}{x-b}$. *Ans.* x^2+b^2.

6. Divide $\frac{2ax+x^2}{a^3-x^3}$ by $\frac{x}{a-x}$. *Ans.* $\frac{2a+x}{a^2+ax+x^2}$.

7. Divide $\frac{14x-3}{5}$ by $\frac{10x-4}{25}$. *Ans.* $\frac{70x-15}{10x-4}$.

8. Divide $\frac{9x^2-3x}{5}$ by $\frac{x^2}{5}$. *Ans.* $\frac{9x-3}{x}$.

9. Divide $\frac{6x-7}{x+1}$ by $\frac{x-1}{3}$. *Ans.* $\frac{18x-21}{x^2-1}$.

10. Divide $\frac{x+x^2}{3a^2}$ by $\frac{2ax+2ax^2}{7}$. *Ans.* $\frac{7}{6a^3}$.

11. Divide $\frac{a^6-x^6}{a^2-2ax+x^2}$ by $\frac{a^2+ax+x^2}{a-x}$. *Ans.* a^3+x^3.

12. Divide $\frac{9y^2-3y}{5}$ by $\frac{y^2}{5}$. *Ans.* $\frac{9y-3}{y}$.

13. Divide $\frac{na-nx}{a+b}$ by $\frac{ma-mx}{a+b}$. *Ans.* $\frac{n}{m}$.

14. Divide a by $\frac{x}{x+y}\times\frac{a}{x-y}$. *Ans.* $\frac{x^2-y^2}{x}$.

15. Divide $\frac{3(x^2-1)}{2(a+b)}$ by $\left(\frac{x+1}{2a}\right)\left(\frac{x-1}{a+b}\right)$. *Ans.* $3a$.

16. Divide $\frac{10ab+3a^2+3b^2}{10ab-3a^2-3b^2}$ by $\frac{3a+b}{b-3a}\cdot\frac{b}{a}$. *Ans.* $\frac{3ab+a^2}{ab-3b^2}$.

17. Divide $\frac{a^2}{x^3}+\frac{1}{a}$ by $\frac{a}{x^2}-\frac{1}{x}+\frac{1}{a}$. *Ans.* $\frac{a+x}{x}$.

18. Divide $\frac{a-1}{a}+\frac{b-1}{b}+\frac{c-1}{c}-1$ by $2-\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)$.

Ans. 1.

REDUCTION OF COMPLEX FORMS.

134. A fraction is said to be *simple*, when both numerator and denominator are entire; otherwise it is said to be *complex*.

135. To reduce a complex to a simple fraction, we may regard the quantity above the line as a dividend, and the quantity below it as a divisor, and proceed according to the last rule.

A more convenient method may be derived from the following observations:

1. If a fraction be multiplied by its own denominator, the product will be the numerator.
2. If a fraction be multiplied by any multiple of its denominator, the product must be entire.

Hence, to simplify a complex fraction, we have the following

RULE.—*Multiply both numerator and denominator by the least common multiple of the denominators of the fractional parts.*

EXAMPLES FOR PRACTICE.

1. Simplify $\dfrac{\dfrac{a}{a-x}-1}{1-\dfrac{a}{a+x}}$.

Multiplying both numerator and denominator by $(a-x)(a+x)$, or by its equal a^2-x^2, we have

$$\frac{\dfrac{a}{a-x}-1}{1-\dfrac{a}{a+x}}=\frac{a^2+ax-a^2+x^2}{a^2-x^2-a^2+ax}=\frac{ax+x^2}{ax-x^2}=\frac{a+x}{a-x}, \textit{ Ans.}$$

2. Simplify $\dfrac{a+\dfrac{b}{c}}{a+\dfrac{c}{b}}$. *Ans.* $\dfrac{abc+b^2}{abc+c^2}$.

3. Simplify $\dfrac{\dfrac{a^3}{bc^2} + \dfrac{b^3}{a^2c}}{\dfrac{a^2}{b^2c} + \dfrac{b^2}{ac^2}}$. *Ans.* $\dfrac{a^4b + b^4c}{a^4c + b^4a}$.

4. Simplify $\dfrac{\dfrac{x-1}{m} - \dfrac{x+1}{n}}{\dfrac{x+1}{m} + \dfrac{x-1}{n}}$. *Ans.* $\dfrac{x(n-m) - (n+m)}{x(n+m) + (n-m)}$.

5. Simplify $\dfrac{\dfrac{a+1}{b} - 2 + \dfrac{b-1}{a}}{\dfrac{a-1}{b} - 2 + \dfrac{b+1}{a}}$. *Ans.* $\dfrac{a-b+1}{a-b-1}$.

6. Simplify $\dfrac{\dfrac{a}{bc} + \dfrac{b}{ac} + \dfrac{c}{ab}}{\dfrac{ab}{c} + \dfrac{ac}{b} + \dfrac{bc}{a}}$. *Ans.* $\dfrac{a^2 + b^2 + c^2}{a^2b^2 + a^2c^2 + b^2c^2}$.

7. Simplify $\dfrac{\dfrac{a+b}{c+d} + \dfrac{a-b}{c-d}}{\dfrac{a+b}{c-d} + \dfrac{a-b}{c+d}}$. *Ans.* $\dfrac{ac - bd}{ac + bd}$.

8. Simplify $\dfrac{\dfrac{a^2+b^2}{a^2-b^2} - \dfrac{a^2-b^2}{a^2+b^2}}{\dfrac{a+b}{a-b} - \dfrac{a-b}{a+b}}$. *Ans.* $\dfrac{ab}{a^2+b^2}$.

9. Simplify $\dfrac{\dfrac{1}{x-1} + \dfrac{1}{x+1}}{\dfrac{1}{y-1} + \dfrac{1}{y+1}}$. *Ans.* $\dfrac{x(y^2-1)}{y(x^2-1)}$.

10. Simplify $\dfrac{\dfrac{a+1}{a} + \dfrac{b+1}{b} - \dfrac{c+1}{c} - \dfrac{d+1}{d}}{\dfrac{cd}{c+d} - \dfrac{ab}{a+b}}$.

Ans. $\dfrac{(a+b)(c+d)}{abcd}$.

SECTION II.

SIMPLE EQUATIONS.

136. An ***Equation*** is an expression of equality between two quantities. Thus,

$$x + y = a$$

is an equation, signifying that the sum of x and y is equal to a.

137. The ***First Member*** of an equation is the quantity on the left of the sign of equality; and

The ***Second Member*** is the quantity on the right of the sign of equality. Thus, in the equation,

$$x - 3y = a - b,$$

the first member is $x - 3y$, and the second member is $a - b$. The two members are sometimes called the two *sides* of the equation.

138. It is important to observe that the kind of equality subsisting in an equation is *algebraic;* that is, the two members must have the *same essential sign,* as well as the same arithmetical value.

139. The ***Unknown Quantity*** of an equation is the letter to which some particular value or values must be given, in order that the statement contained in the equation may be true. And such value or values are said to *satisfy the equation.* An equation may contain two or more unknown quantities.

140. A ***Root*** of an equation is any value which, being substituted for the unknown quantity, will satisfy the equation. For example, in

$$x^2 + 2x = 35$$

let us substitute 5 for x; we shall then have

$$5^2 + 2 \times 5 = 35,$$
$$25 + 10 = 35,$$
$$35 = 35.$$

Hence, 5 is a *root* of the equation, because if substituted for x, it will render the two members equal. Again, let $x = -7$. We have

$$(-7)^2 + (-7 \times 2) = 35,$$
$$49 - 14 = 35,$$
$$35 = 35.$$

Hence, -7 is also a root of the given equation.

141. A ***Numerical Equation*** is one in which all the known quantities are expressed by figures, as, $3x^3 - x^2 + 2x = 17$.

142. A ***Literal Equation*** is one in which some or all of the known quantities are expressed by letters; as $ax^2 - 3bx = 5d$.

143. An ***Equation of Condition*** is one which must exist between certain known or arbitrary quantities, in order that certain other equations may be true. Thus, the two equations,

$$x + c = 5a,$$
$$x - c = a,$$

cannot both be true at the same time, unless

$$c = 2a;$$

that is, the last equation expresses the condition which will render the other two equations true; it is therefore called an *equation of condition.*

144. An ***Identical Equation*** is one in which the two members are the same algebraic expression, or are reducible to the same. Thus,

$$a^2 - 3x = a^2 - 3x,$$
$$x^2 - a^2 = (x + a)(x - a),$$

are identical equations.

145. Equations are said to be of different degrees or dimensions.

The ***Degree*** of an equation is denoted by the greatest number of unknown factors occurring in any term. Hence,

1. If an equation involves but one unknown quantity, its degree is denoted by the highest exponent of this quantity in any term.

2. If an equation involves more than one unknown quantity, its degree is denoted by the greatest sum which the exponents of the unknown quantities give in any term.

Thus, for example:

$$\left.\begin{aligned} x + ax &= b \\ ax + y &= c^2 \end{aligned}\right\} \text{are equations of the first degree;}$$

$$\left.\begin{aligned} x^2 + 4x &= 8 \\ x^2 + xy &= a^2b \end{aligned}\right\} \text{are equations of the second degree;}$$

$$\left.\begin{aligned} ax^3 + bx^2 + cx &= 2a^4b \\ x^3 + 3xy + y^3 &= ab^5 \end{aligned}\right\} \text{are equations of the third degree.}$$

146. A ***Simple Equation*** is an equation of the first degree.

147. A ***Quadratic Equation*** is an equation of the second degree.

148. A ***Cubic Equation*** is an equation of the third degree.

TRANSFORMATION OF EQUATIONS.

149. The ***Transformation*** of an equation is the process of changing its form without destroying the equality of its members.

From the nature of an equation, it is evident that all the operations to which it can be subjected without destroying the equality, are embraced in the axioms (**39**); they may be stated as follows:

1. The same or equal quantities may be added to both members (Ax. 1).
2. The same or equal quantities may be subtracted from both members (Ax. 2).
3. Both members may be multiplied by the same or equal quantities (Ax. 3).
4. Both members may be divided by the same or equal quantities (Ax. 4).
5. Both members may be raised, by involution, to the same power (Ax. 8).
6. Both members may be reduced, by evolution, to the same root (Ax. 9).

CASE I.

150. To transpose the terms of an equation.

Transposition is the process of changing a term from one member of an equation to the other, without destroying the equality.

To exhibit the law of transposition, let us consider the three following examples:

1. Let $$x + a = b.$$

If we subtract a from both members of this equation, the result will be

$$x = b - a;$$

and we perceive that the term, $+a$, has been removed from the first member, and appears as $-a$ in the second member.

2. Let $$x - a = b.$$

If we add a to both members of this equation, the result will be

$$x = b + a;$$

and we perceive that the term, $-a$, has been removed from the first member, and appears as $+a$ in the second.

3. Let $$a - x = b.$$

Subtracting a from both members of the equation, we have

$$-x = -a + b.$$

If we now multiply both members of this result by -1, we shall have

$$x = a - b;$$

and by comparing this last result with the given equation, we observe that $+a$ has been removed from the first to the second member, but the signs of both the other terms of the equation have been changed.

Hence, for changing the sign or place of any term of an equation we have the following

RULE.—I. *Any term may be transposed from one member of an equation to the other by changing its sign* (1, 2).

II. *Any term may be transposed without changing its sign, provided the signs of all the other terms be changed* (3).

III. *The sign of any term may be changed without transposition, by changing the signs of all the terms simultaneously* (3).

EXAMPLES FOR PRACTICE.

In the following equations, transpose the unknown terms to the first member, and the known terms to the second (I):

1. $a^2x + bc = ab - 2ax$. *Ans.* $a^2x + 2ax = ab - bc$.

2. $3b^3 - 2x - 5 = 3c - 5ax - dx$.
Ans. $5ax + dx - 2x = 3c - 3b^3 + 5$.

3. $4c^2x - a + 3b = x - ab - 2cx$.
Ans. $4c^2x - x + 2cx = a - 3b - ab$.

4. $5ab^2 - x + 4cd = ax - cx + a^3$.
Ans. $cx - ax - x = a^3 - 5ab^2 - 4cd$.

In the following, transpose the unknown terms to the first member, and the known to the second (II)

5. $ax + bc = a^3 + c^2x$. *Ans.* $c^2x - ax = bc - a^3$.

6. $4cd^2 - a^2x - 3cm = ax - m^2$.
Ans. $a^2x + ax = 4cd^2 - 3cm + m^2$.

7. $ax - 7 + 5cd = bc + a^2cx - 4m^2$.
Ans. $a^2cx - ax = 5cd - 7 - bc + 4m^2$.

8. $a^3 - c^2x - 3dx = c^2d^2x - 5b^2$.
Ans. $c^2d^2x + c^2x + 3dx = a^3 + 5b^2$.

CASE II.

151. To clear an equation of fractions.

We have seen (**135,** 2), that if a fraction be multiplied by any multiple of its denominator, the product will be entire; consequently, if several fractions be multiplied by a *common multiple* of their denominators, all the products will be entire.

Let us take the equation,

$$\frac{3x}{10} - \frac{2x}{15} = 12.$$

Multiplying every term by 30, which is the *least common multiple* of the denominators, we have

$$9x - 4x = 360,$$

in which all the terms are entire.

Again, let $$\frac{x}{a^2} - \frac{x-c}{ab^2} = \frac{x+c}{a^2b}.$$

Multiplying every term by a^2b^2, observing that the product obtained from the second fraction is to be *subtracted*, we have

$$b^2x - ax + ac = bx + bc.$$

Hence the following

RULE.—*Multiply all the terms of the equation by the least common multiple of the denominators, observing that when a fraction has the minus sign before it, the signs of the terms derived from its numerator must be changed.*

NOTES.—1. The pupil should observe that in multiplying any fraction it will be most convenient to divide the multiplier by the denominator and multiply the numerator by the quotient.

2. It will be obvious, also, that the equation will be cleared of fractions, by multiplying by the several denominators, successively.

EXAMPLES FOR PRACTICE.

Clear the following equations of fractions:

1. $\frac{x}{2} + \frac{2x}{3} - \frac{3x}{4} = 10.$ *Ans.* $6x + 8x - 9x = 120.$

2. $\frac{3x}{7} - \frac{2x+3}{14} = \frac{x-5}{21}.$ *Ans.* $18x - 6x - 9 = 2x - 10.$

3. $\frac{a}{x-a} + \frac{c}{x+a} = \frac{d}{x^2-a^2}.$ *Ans.* $ax + a^2 + cx - ac = d.$

4. $\frac{x-a}{c} - \frac{2x-3a}{ac^2} = \frac{x+ac}{a^2}.$
Ans. $a^2cx - a^3c - 2ax + 3a^2 = c^2x + ac^3.$

5. $\frac{ax-bx}{8c} - \frac{cx-ax}{10a} = \frac{bx-cx}{4ac}.$
Ans. $5a^2x - 5abx - 4c^2x + 4acx = 10bx - 10cx.$

6. $\frac{5x}{12} - \frac{3x}{16} + \frac{3-x}{24} - \frac{5x-2}{20} = 2.$
Ans. $100x - 45x + 30 - 10x - 60x + 24 = 480.$

7. $\frac{1}{abc} = \frac{a}{bcx} + \frac{b}{acx} + \frac{c}{abx}.$ *Ans.* $x = a^2 + b^2 + c^2.$

SOLUTION OF SIMPLE EQUATIONS.

152. The ***Solution of an Equation*** is the process of finding the value or values of the unknown quantity, or the roots of the equation.

153. A root of an equation is said to be *verified,* if the two members of the equation prove to be equal after the root has been substituted for the unknown quantity.

154. A simple equation may be solved, by transforming it in such a manner that the unknown quantity shall stand alone, and constitute one member of the equation; the other member will then be the value of the unknown quantity, or the root of the equation.

Let it be required to find the value of x in the equation

$$\frac{5x-2}{3}-\frac{x-7}{4}=4+\frac{5x}{6}\ldots(1).$$

Clearing of fractions,

$$20x-8-3x+21=48+10x\ldots(2).$$

By transposition,

$$20x-3x-10x=48+8-21\ldots(3).$$

Uniting similar terms, $\qquad 7x=35\ldots(4).$

Dividing both members by 7, $\qquad x=5\ldots(5).$

To *verify* this value of x, substitute it for x in equation (1); we shall have

$$\frac{25-2}{3}-\frac{5-7}{4}=4+\frac{25}{6}.$$

Reducing each term to its simplest form, we obtain

$$7\tfrac{2}{3}+\frac{1}{2}=4+4\tfrac{1}{6};$$

whence, by addition, we have

$$8\tfrac{1}{6}=8\tfrac{1}{6};$$

the value of x is therefore verified.

155. It should here be observed that an equation of the first degree, containing but one unknown quantity, cannot have more than one root. For, whatever the equation may be, suppose it to

be cleared of fractions, and the unknown terms transposed to the first member, and the known terms to the second. Then if we represent the algebraic sum of the coefficients of x by a, and the second member by b, the equation will take this general form:

$$ax = b \ldots (1).$$

Now, if possible, suppose that this equation has two roots, r and r'. Then since every root must satisfy the equation (**140**), we shall have, by substituting r and r' successively in (1),

$$ar = b \ldots (2),$$
$$ar' = b \ldots (3);$$

whence, by Ax. 7, we shall have

$$ar = ar' \ldots (4);$$

or, by transposing and factoring,

$$a(r - r') = 0 \ldots (5).$$

But equation (5) is impossible, since, by supposition, $r - r'$ is not zero, and a is not zero. Hence,

An equation of the first degree cannot have more than one root.

156. From these principles and illustrations we derive the following

RULE.—I. *If necessary, clear the equation of fractions, and perform all the operations indicated.*

II. *Transpose the unknown terms to the first member and the known terms to the second, and reduce each member to its simplest form, factoring, when necessary, with reference to the unknown quantity.*

III. *Divide both members by the coefficient of the unknown quantity, and the second member will be the value required, or the root of the equation.*

The three principal steps in the reduction of a simple equation, containing but one unknown quantity, may be briefly stated as follows:

1st. Clearing of fractions.

2d. Transposing and uniting terms.

3d. Dividing by the coefficient of x.

PRACTICAL SUGGESTIONS.

There are certain cases in which the preceding rule may be modified, with advantage, by special artifices.

1. When the equation contains similar terms, or fractions having a common denominator, these should be united as far as possible before clearing of fractions. Thus,

Given $$56 + \frac{7x - 10}{4} + \frac{x - 7}{7} = 100 - \frac{x + 10}{4};$$

transposing and uniting terms, $$2x + \frac{x - 7}{7} = 44;$$

clearing of fractions, $$14x + x - 7 = 308;$$
$$15x = 315;$$
$$x = 21.$$

2. When the equation contains fractions whose numerators or denominators are polynomial, we may clear the equation of its simpler denominators first, uniting the entire quantities at each step, if possible. Thus,

Given $$\frac{6x + 7}{9} + \frac{7x - 13}{6x + 3} = \frac{2x + 4}{3};$$

multiplying by 9, $$6x + 7 + \frac{21x - 39}{2x + 1} = 6x + 12;$$

transposing and uniting, $$\frac{21x - 39}{2x + 1} = 5;$$

clearing of fractions, $$21x - 39 = 10x + 5;$$
$$11x = 44;$$

whence, by division, $$x = 4.$$

3. When the equation contains but a single numerical term, we may simply *indicate* the multiplication of this term, in the clearing of fractions, until the final step in the solution is reached. Thus,

Given $$\frac{x}{4} + \frac{x}{7} + \frac{x}{12} + \frac{x}{21} = 88;$$

multiplying by 84, $$21x + 12x + 7x + 4x = 88 \times 84;$$
$$44x = 88 \times 84;$$

dividing by 44, $$x = 2 \times 84;$$
$$x = 168.$$

EXAMPLES FOR PRACTICE.

Find the value of x in each of the following equations:

1. $7x - 16 = 3x - 4.$ *Ans.* $x = 3.$
2. $3x + 9 = 5x + 1.$ *Ans.* $x = 4.$
3. $4x + 7 = x + 21 - 3 + x.$ *Ans.* $x = 5\frac{1}{2}.$
4. $5x + 16 = x + 52.$ *Ans.* $x = 9.$
5. $5ax - c = b - 3ax.$ *Ans.* $x = \frac{b+c}{8a}.$
6. $ax + b = 9x + c.$ *Ans.* $x = \frac{c-b}{a-9}.$
7. $\frac{x}{4} + \frac{x}{6} = 10.$ *Ans.* $x = 24.$
8. $\frac{3x}{2} = \frac{x}{4} + 24.$ *Ans.* $x = 19\frac{1}{5}.$
9. $\frac{3x+5}{2} = \frac{15x-1}{8}.$ *Ans.* $x = 7.$
10. $\frac{x+1}{3} + \frac{3x-5}{5} = \frac{9x}{10}.$ *Ans.* $x = 20.$
11. $\frac{2x+1}{2} + \frac{7x-15}{5} = \frac{17x+3}{8} - \frac{3}{2}.$ *Ans.* $x = 5.$
12. $\frac{x}{2} + \frac{x}{3} + \frac{5x}{12} = \frac{5x}{7} + \frac{3x}{4} - 18.$ *Ans.* $x = 84.$
13. $\frac{17x-12}{3} - \frac{5x-16}{4} - \frac{10x-3}{6} = \frac{6x-7}{2}.$ *Ans.* $x = 16.$
14. $21 + \frac{3x-11}{16} = \frac{5x-5}{8} + \frac{97-7x}{2}.$ *Ans.* $x = 9.$
15. $\frac{7x+16}{21} - \frac{x+8}{4x-11} = \frac{x}{3}.$ *Ans.* $x = 8.$
16. $\frac{9x+20}{36} = \frac{4x-12}{5x-4} + \frac{x}{4}.$ *Ans.* $x = 8.$
17. $\frac{20x}{25} + \frac{36}{25} + \frac{5x+20}{9x-16} = \frac{4x}{5} + \frac{86}{25}.$ *Ans.* $x = 4.$

18. $\frac{3x}{4} - \frac{x-1}{2} = 6x - \frac{20x+13}{4}$. *Ans.* $x = 5$.

19. $\frac{x-3}{2} + \frac{x}{3} = 20 - \frac{x+19}{2}$. *Ans.* $x = 9$.

20. $\frac{x+1}{2} + \frac{x+2}{3} = 16 - \frac{x+3}{4}$. *Ans.* $x = 13$.

21. $2x - \frac{x+3}{3} + 15 = \frac{12x+26}{5}$. *Ans.* $x = 12$.

22. $\frac{5x+5}{x+2} - 29 = \frac{6x-12}{x-2} - 30$. *Ans.* $x = 2$.

23. $\frac{7x+9}{4} - \left(x - \frac{2x-1}{9}\right) = 7$. *Ans.* $x = 5$.

24. $\frac{7+9x}{4} - \left(1 - \frac{2-x}{9}\right) = 7x$. *Ans.* $x = \frac{1}{5}$.

25. $\frac{x+1}{2} + \frac{x+2}{3} = \frac{x-3}{4} + \frac{x-4}{6} + 3$. *Ans.* $x = 1$.

26. $\frac{x}{2} + \frac{x}{3} + \frac{x}{4} + \frac{x}{5} = 77$. *Ans.* $x = 60$.

27. $\frac{x}{2} + \frac{x}{3} + \frac{x}{4} = 130$. *Ans.* $x = 120$.

28. $\frac{x}{2} + \frac{x}{6} + \frac{x}{12} = 90$. *Ans.* $x = 120$.

29. $\frac{x}{2} + \frac{x}{3} + \frac{x}{7} = 82$. *Ans.* $x = 84$.

30. $\frac{x}{5} + \frac{x}{7} + \frac{x}{12} + \frac{x}{20} + \frac{x}{21} = 660$. *Ans.* $x = 1260$.

31. $a^2x + 2ac - c^2x = a^2 + c^2$. *Ans.* $x = \frac{a-c}{a+c}$.

32. $4bx - 2a = 3ab - 6b^2x$. *Ans.* $x = \frac{a}{2b}$.

33. $a(x-b) + b(x-c) + c(x-a) = 0$. *Ans.* $x = \frac{ab+ac+bc}{a+b+c}$.

34. $a^2(x-1)+am(x-2)=m^2.$ *Ans.* $x=\dfrac{a+m}{a}.$

35. $ax+cx+x=b+\dfrac{b-ax}{c}.$ *Ans.* $x=\dfrac{b}{a+c}.$

36. $\dfrac{a+x}{b}+\dfrac{c-x}{d}=\dfrac{a}{b}.$ *Ans.* $x=\dfrac{bc}{b-d}.$

37. $\dfrac{x}{a-1}+\dfrac{x}{b-1}-\dfrac{x}{a+1}-\dfrac{x}{b+1}=1.$

Ans. $x=\dfrac{(a^2-1)(b^2-1)}{2(a^2-2+b^2)}.$

38. $\dfrac{x-1}{c-1}+\dfrac{x}{c+1}=\dfrac{1}{c-1}+\dfrac{2}{(c-1)^2}.$ *Ans.* $x=\dfrac{c+1}{c-1}.$

39. $\dfrac{x}{a}+\dfrac{x}{b}+\dfrac{x}{c}=ab+ac+bc.$ *Ans.* $x=abc.$

40. $\dfrac{x-b-c}{a}+\dfrac{x-a-c}{b}+\dfrac{x-a-b}{c}=3.$

Ans. $x=a+b+c.$

41. $1.25x-6.125+.25x=.625x.$ *Ans.* $x=7.$

42. $3.164x-4.266=.24x+.08x.$ *Ans.* $x=1.5.$

43. $\dfrac{2.4x-.12}{2.8}+\dfrac{4.6x-3.6}{4}=\dfrac{.64x-.048}{.7}.$ *Ans.* $x=.8.$

PROBLEMS

PRODUCING EQUATIONS WHICH CONTAIN ONE UNKNOWN QUANTITY.

157. A ***Problem***, in Algebra, is a question requiring the values of one or more unknown quantities from given conditions.

158. The ***Solution*** of a problem is the process of finding the values of the unknown quantities.

159. Every problem in Algebra contains a statement of the relations between certain known and certain unknown quantities. When these relations are such as to furnish one or more equalities, the process of solution consists in expressing these equalities algebraically, and in solving the equations thus obtained.

160. There are two classes of problems which may be solved by the use of a single equation.

1st. Questions referring to a single unknown quantity.

2d. Questions referring to two or more unknown quantities, so related that when one is known, the others may be determined directly by the given conditions.

The following are examples of the first class:

1. What number is that the sum of whose third and fourth parts is 21?

Let x represent the number; then by the conditions,

$$\frac{x}{3} + \frac{x}{4} = 21;$$

clearing of fractions, $4x + 3x = 21 \times 12,$

$$7x = 21 \times 12,$$

whence, $x = 36,$ *Ans.*

2. A and B have each the same annual income. A's yearly expenses are \$800 and B's \$1000, and A saves as much in 5 years as B saves in 7 years; how much is the annual income?

Let $x =$ the income;

then $x - 800 =$ A's annual savings;

and $x - 1000 =$ B's " " .

Now by the conditions of the problem, we have

$$5(x - 800) = 7(x - 1000);$$

whence, $5x - 4000 = 7x - 7000,$

$$2x = 3000,$$

$$x = 1500, \text{ Ans.}$$

The following are examples of the second class:

3. Three men form a copartnership with a joint capital of \$7200. A put in a certain sum, B put in three times as much as A, and C put in as much as both A and B; how much did each man furnish?

Let $x =$ A's share;

then $3x =$ B's " ,

and $4x =$ C's " .

By the conditions, $8x = \$7200;$

whence, $x = \$900,$ A's share,

$3x = \$2700,$ B's share,

$4x = \$3600,$ C's share.

4. There are two numbers whose difference is 6; and if $\frac{1}{5}$ of the less be added to $\frac{1}{3}$ of the greater, the sum will be equal to $\frac{1}{5}$ of the greater diminished by $\frac{1}{5}$ of the less. Required the two numbers.

Let $x =$ the less; then $x + 6 =$ the greater.

By the conditions of the problem,

$$\frac{x}{3} + \frac{x+6}{5} = \frac{x+6}{3} - \frac{x}{5};$$

clearing of fractions, $5x + 3x + 18 = 5x + 30 - 3x$;

whence, $6x = 12$;

$x = 2$, the less,

$x + 6 = 8$, the greater.

These examples illustrate the three essential steps in the solution of any problem requiring but one equation; and we may derive from them the following

GENERAL RULE.

I. *Represent one of the unknown quantities by some letter or symbol, and then from the given relations find an algebraic expression for each of the other unknown quantities, if any, involved in the question.*

II. *Form an equation from some condition, expressed or implied, by indicating the operations necessary to verify the value of the unknown quantity represented by the symbol.*

III. *Solve the equation thus derived.*

The three steps in this process may be named as follows:

1st. The notation.

2d. The equation.

3d. The solution of the equation.

REMARKS.

1. By the first two steps, the conditions of the problem are translated from *common* into *algebraic* language. This is called the *statement* of the question.

2. The chief difficulty in the solution of a problem is generally experienced in obtaining the statement. This arises in part from

the fact that among the problems which may be proposed, there exists an indefinite variety of conditions; and the operator is left very much to his ingenuity, both in adopting suitable notation for any problem, and in deriving the equation.

3. Algebraic problems present two kinds of conditions, *Explicit* and *Implicit.* An explicit condition is one which is distinctly and formally expressed in the language of the problem. An implicit condition is one which is not directly expressed, but only implied, or left to be inferred from other conditions.

4. In any determinate problem there are as many conditions as there are quantities to be determined. And if we represent one of the unknown quantities by an arbitrary symbol, and then proceed to derive expressions for the other unknown quantities, if any, each from a separate condition, there will always remain a final condition, either explicit or implicit, from which to derive the equation.

PROBLEMS FOR SOLUTION.

1. What number is that from which if 6 be subtracted, and the remainder multiplied by 11, the product will be 121?
Ans. 17.

2. A man holds a lease for 20 years, and $\frac{1}{3}$ of the time past is equal to $\frac{1}{2}$ of the time to come; how much of the time has passed?
Ans. 12 years.

3. What number is that which being increased by $\frac{1}{2}$, $\frac{1}{3}$, and $\frac{1}{4}$ of itself is equal to 250? *Ans.* 120.

4. Divide 77 into two such parts that if one part be divided by 7 and the other by 3, the sum of the quotients will be 15.
Ans. 56 and 21.

5. The sum of two numbers is 75, and their difference is equal to $\frac{1}{3}$ of the greater; what are the numbers? *Ans.* 45 and 30.

6. After paying away $\frac{1}{4}$ and $\frac{1}{5}$ of my money, I had \$66 left; how much had I at first? *Ans.* \$120.

7. After paying away $\frac{1}{3}$ of my money, and $\frac{1}{4}$ of what remained, and losing $\frac{1}{5}$ of what was then left, I still had \$24; how much had I at first? *Ans.* \$60.

8. What number is that from which if 5 be subtracted, $\frac{2}{3}$ of the remainder will be 40? *Ans.* 65.

9. A man sold a horse and a chaise for \$200, and $\frac{1}{4}$ of the price of the horse was equal to $\frac{1}{6}$ of the price of the chaise. What was the price of each? *Ans.* Chaise, \$120; horse, \$80.

10. Divide 48 into two such parts, that if the less be divided by 4, and the greater by 6, the sum of the quotients will be 9. *Ans.* 12 and 36.

11. An estate was divided among 4 children in the following manner: The first received \$200 more than $\frac{1}{4}$ of the whole, the second \$340 more than $\frac{1}{5}$ of the whole, the third \$300 more than $\frac{1}{6}$ of the whole, and the fourth \$400 more than $\frac{1}{8}$ of the whole; what was the value of the estate? *Ans.* \$4800.

12. What number is that from which if 91 be subtracted, $\frac{1}{3}$ of the remainder will be equal to $\frac{1}{10}$ of the number? *Ans.* 130.

13. Four men take stock in a railroad company, amounting in the aggregate to \$73500. A takes a certain sum, B takes three times as much as A, C takes three times as much as A and B together, and D takes $\frac{1}{3}$ as much as B and C. How much of the stock does A take? *Ans.* \$3500.

14. Divide 105 into two parts which shall be to each other as 3 to 4.

Since the parts are to each other as 3 to 4, let

$3x =$ the less part; then $4x =$ the greater;

hence $7x = 105, \quad x = 15;$

$3x = 45$, the less;

$4x = 60$, the greater.

15. A and B shared between themselves a bequest of \$2000, in the ratio of 7 to 9. How much did each receive? *Ans.* A, \$875; B, \$1125.

16. A farmer made a mixture of rye, oats, and peas, using 3 bushels of rye as often as 4 of oats and 5 of peas. The whole amount of grain used was 72 bushels; how many bushels were there of each kind?

Ans. Rye, 18 bushels; oats, 24 bushels; peas, 30 bushels.

17. From two casks of equal size were drawn quantities which are in the ratio of 6 to 7; and it appears that if 16 gallons less had been drawn from that which now contains the less, only one half as much would have been drawn from it as from the other. How many gallons were drawn from each? *Ans.* 24 and 28.

18. It is required to divide the number 204 into two such parts that $\frac{2}{5}$ of the less being taken from the greater, the remainder will be equal to $\frac{3}{7}$ of the greater subtracted from 4 times the less.

Ans. 154 and 50.

19. A man bought a horse and chaise for 341 dollars. Now if $\frac{3}{8}$ of the price of the horse be subtracted from twice the price of the chaise, the remainder will be the same as if $\frac{5}{7}$ of the price of the chaise be subtracted from 3 times the price of the horse. Required the price of each. *Ans.* Horse, \$152; chaise, \$189.

20. A certain sum of money was put at simple interest; in 8 months it amounted to \$1488, and in 15 months it amounted to \$1530. What was the sum at interest?

It is often convenient, in the solution of a problem, to avoid the multiplication of large numerals. This may be done by representing a given number by a letter, as follows:

Put $a = 1488$; then $a + 42 = 1530$.

Let $x =$ the sum at interest;

then $a - x =$ interest for 8 months;

and $a + 42 - x =$ " " 15 "

Equating two expressions for the monthly interest,

$$\frac{a-x}{8} = \frac{a+42-x}{15};$$

whence, $15a - 15x = 8a + 336 - 8x,$

$$7x = 7a - 336,$$

and $x = a - 48 = \$1440$, *Ans.*

21. A prize having been captured by a privateer, the sum of \$7560 was awarded to the officers, and the residue was divided equally among the crew, consisting of 27 men. If the officers had received \$9560, and the crew had consisted of 25 men, each private would have received the same sum for his share; what was the value of the prize? *Ans.* \$34560.

22. A merchant allows \$1000 per annum for the expenses of his family, and annually increases that part of his capital which is not so expended by a third of it; at the end of three years his original capital is doubled. What had he at first?

Ans. \$14800.

23. A man having a lease for 99 years, was asked how much of it had already expired; he answered that $\frac{2}{3}$ of the time past was equal to $\frac{4}{5}$ of the time to come. Required the time past and the time to come.

Ans. Time past, 54 years; time to come, 45 years.

24. In the composition of a quantity of gunpowder, the nitre was 10 pounds more than $\frac{2}{3}$ of the whole, the sulphur was $4\frac{1}{2}$ pounds less than $\frac{1}{6}$ of the whole, and the charcoal was 2 pounds less than $\frac{1}{7}$ of the nitre. What was the amount of the gunpowder? *Ans.* 69 pounds.

25. Divide \$183 between two men, so that $\frac{4}{7}$ of what the first receives shall be equal to $\frac{3}{10}$ of what the second receives.

Ans. 1st, \$63; 2d, \$120.

26. Divide the number 68 into two such parts, that the difference between the greater and 84 shall be equal to 3 times the difference between the less and 40. *Ans.* Greater, 42; less, 26.

27. Four places are situated in the order of the letters, A, B, C, D. The distance from A to D is 34 miles; the distance from A to B is to the distance from C to D as 2 to 3; and $\frac{1}{4}$ of the distance from A to B, added to one half the distance from C to D, is three times the distance from B to C. What are the respective distances? *Ans.* 12, 4, and 18 miles.

28. A man driving a flock of sheep to market, was met by a party of soldiers, who plundered him of $\frac{1}{3}$ of his flock and 6 more. Afterward he was met by another company, who took $\frac{1}{2}$ what he then had and 10 more; after that he had but 2 left. How many had he at first? *Ans.* 45.

29. A boy engaged to carry 100 glass vessels to a certain place, on the condition that he should receive 3 cents for every one he delivered, and forfeit 9 cents for every one he broke. On settlement, he received 240 cents; how many vessels did he break?

Ans. 5.

30. A person's entire indebtedness to A, B, and C, was \$270. His indebtedness to B was twice as much as to A, and his indebtedness to C was twice as much as to A and B. How much did he owe each? *Ans* A, \$30; B, \$60; C, \$180.

31. A company of 4 laborers received \$315. B received $1\frac{1}{2}$ times as much as A, C received $1\frac{1}{3}$ times as much as A and B, and D received $1\frac{1}{4}$ times as much as A, B, and C. What did each laborer receive? *Ans.* A, \$24; B, \$36; C, \$80; D, \$175.

NOTE.—Let $6x$ represent A's share, and $9x$ B's share.

32. A gamester, after losing $\frac{1}{5}$ of his money, won 4 shillings; he then lost $\frac{1}{4}$ of what he had, and afterward won 3 shillings; he then lost $\frac{1}{6}$ of what he had, and found that he had only 20 shillings remaining. How much had he at first? *Ans.* 30 shillings.

33. A gentleman spends $\frac{2}{3}$ of his yearly income for the support of his family, and $\frac{2}{3}$ of the remainder in improvements on his premises, and lays by \$70 a year. What is his income?

Ans. \$630.

34. Divide the number 60 into two such parts that the product of the two parts may be equal to 3 times the square of the less part. *Ans.* 15 and 45.

35. My horse and saddle are together worth 90 dollars, and my horse is worth 8 times as much as my saddle. What is the value of each? *Ans.* Saddle, \$10; horse, 80.

36. Divide \$462 between two persons, so that for every dime which one receives, the other may receive a dollar.

Ans. \$42 and \$420.

37. The rent of an estate is 8 per cent. greater this year than last. This year it is 1890 dollars; what was it last year?

Ans. \$1750.

38. The sum of two numbers is 840, and their difference is equal to $\frac{1}{3}$ of the greater. What are the numbers?

Ans. 504 and 336.

39. A person, after spending 100 dollars more than $\frac{1}{5}$ of his income, had remaining 35 dollars more than $\frac{1}{2}$ of it. Required his income. *Ans.* \$450.

40. Divide \$1520 among A, B, and C, so that B shall have \$100 more than A, and C \$270 more than B.

Ans. A, \$350; B, \$450; C, \$720.

41. A and B have the same income. A contracts an annual debt amounting to $\frac{1}{7}$ of it; B lives upon $\frac{4}{5}$ of it; at the end of two years B lends to A enough to pay off his debts, and has 32 dollars to spare. What is the income of each? *Ans.* \$280.

42. A sets out from a certain place, and travels at the rate of 7 miles in 5 hours; and 8 hours afterward B sets out from the same place in pursuit, at the rate of 5 miles in 3 hours. How long and how far must B travel before he overtakes A?

Ans. 42 hours; 70 miles.

43. A can perform a certain piece of work in 8 days, and B can do the same in 12 days; in how many days can both, working together, do it? *Ans.* $4\frac{4}{5}$.

44. A person has just 6 hours at his disposal; how far may he ride in a coach which travels 8 miles an hour, that he may return home in time, walking back at the rate of 4 miles an hour?

Ans. 16 miles.

45. A can dig a trench in one half the time required by B, B can dig it in two thirds of the time required by C, and all together can dig it in 6 days; find the time that each alone would require. *Ans.* A, 11 days; B, 22 days; C, 33 days.

46. A and B start from opposite points and travel toward each other, A at the rate of 3 miles an hour, and B at the rate of 4 miles an hour. At the same time, C sets out with A and travels at the rate of 5 miles an hour. After meeting B he turns back and travels until he meets A; he then finds that the whole time elapsed since starting is 10 hours. How far apart were A and B at the beginning? *Ans.* 72 miles.

47. Two farmers owning a flock of sheep, agree to divide them. A takes 72 sheep; B takes 92 sheep, and pays A \$35. Required the value of the flock. *Ans.* \$574.

48. A crew which can row at the rate of 12 miles an hour in still water, finds that it takes 7 hours to come up a river a certain distance, and 5 hours to go down again. At what rate does the river flow? *Ans.* 2 miles per hour.

SIMPLE EQUATIONS

CONTAINING TWO UNKNOWN QUANTITIES.

161. We have seen that every simple equation containing only one unknown quantity can be satisfied with one value, and only one value, of the unknown quantity (**155**). But if we consider a single equation containing two unknown quantities, we shall find that for every value which we please to give to one of the unknown quantities, we can determine a corresponding value of the other unknown quantity, such that the *set* of values will satisfy the equation.

Thus, let $$2x + 3y = 17 \ldots (1).$$

Put $x = 1$, and substitute this value in the given equation; we have
$$2 + 3y = 17,$$
$$y = 5.$$

Now the set of values, $x = 1$, $y = 5$, will satisfy the equation; for, by substitution, we have
$$2 + 15 = 17.$$

In the same manner, we may obtain the following sets of values, each one of which will satisfy equation (1):

1.	$x = 1,$	$y = 5.$
2.	$x = 2,$	$y = 4\frac{1}{3}.$
3.	$x = 3,$	$y = 3\frac{2}{3}.$
4.	$x = 4,$	$y = 3.$

It is evident that there is no limit to the number of sets of values that may be obtained. The equation, and also the quantities, in such cases, are said to be *indeterminate.* Hence,

162. An ***Indeterminate Equation*** is one which is satisfied by an infinite number of values of the unknown quantities.

A single equation containing two unknown quantities is indeterminate.

163. If we take two equations with two unknown quantities, as
$$2x + 5y = 31 \ldots (1),$$
$$3x + 2y = 19 \ldots (2),$$
it is evident that we may obtain as many sets of values as we please,

which will satisfy each equation, *considered separately.* Thus, proceeding as before, we find that the set,

$$x = 5, \qquad y = 4\tfrac{1}{5},$$

will satisfy the first equation; and a different set,

$$x = 4, \qquad y = 3\tfrac{1}{2},$$

will satisfy the second equation.

Now suppose we are required to satisfy *both* equations with the *same set* of values for x and y.

Multiplying (1) by 3, and (2) by 2,

$$6x + 15y = 93 \ldots (3),$$

$$6x + 4y = 38 \ldots (4).$$

Subtracting (4) from (3),

$$11y = 55 \ldots (5);$$

whence, $$y = 5 \ldots (6).$$

Substituting this value of y in (1),

$$2x + 25 = 31 \ldots (7);$$

whence, $$x = 3 \ldots (8).$$

Thus we have a single set of values,

$$x = 3, \qquad y = 5,$$

which will satisfy both equations. For, let these values be substituted in the given equations; we shall have

$$6 + 25 = 31,$$

$$9 + 10 = 19.$$

Equations thus related are said to be *simultaneous.* Hence,

164. ***Simultaneous Equations*** are those which must be satisfied by the same values of the unknown quantities which enter them.

When two or more simultaneous equations are given, the values of the unknown quantities are determined by a process called

ELIMINATION.

165. ***Elimination*** is the process of combining equations in such a manner as to cause one or more of the unknown quantities contained in them to disappear.

There are four principal methods of elimination:

1st, *By addition and subtraction;* 2d, *By comparison;* 3d, *By substitution;* 4th, *By indeterminate multipliers.*

CASE I.

166. Elimination by addition and subtraction.

1. Given $3x + 2y = 23$, and $4x - 3y = 8$, to find the values of x and y.

OPERATION.

$$3x + 2y = 23 \ldots (1),$$
$$4x - 3y = 8 \ldots (2).$$

Multiplying (1) by 4, and (2) by 3, $12x + 8y = 92 \ldots (3)$,
$$12x - 9y = 24 \ldots (4);$$
subtracting (4) from (3), $17y = 68 \ldots (5)$;
whence, $y = 4 \ldots (6)$.

Thus, we have eliminated x, and found the value of y.

Again, multiplying (1) by 3, and (2) by 2,

$$9x + 6y = 69 \ldots (7),$$
$$8x - 6y = 16 \ldots (8);$$

adding (7) to (8), $17x = 85$;
whence, $x = 5$.

We have thus eliminated y, and found the value of x. Hence,

RULE.—I. *Multiply or divide the equations by such numbers or quantities that the coefficients of the quantity to be eliminated shall be made equal in the two equations.*

II. *If these coefficients have like signs, subtract one of the prepared equations from the other, member from member; if they have unlike signs, add the equations, member to member.*

NOTES.—1. In preparing the given equations by multiplication, find the least common multiple of the coefficients of the letter to be eliminated, and divide this multiple by each coefficient; the quotients will be the *least multipliers* that can be used. If the coefficients are prime to each other, it is evident that each equation must be multiplied by the coefficient in the other equation.

2. It is generally convenient to clear the equations of fractions, before applying the rule. This is not necessary, however. For if any letter has fractional coefficients in the two equations, the fractions may be reduced to a common denominator; it will then be necessary to render the numerators equal by multiplication or division, according to the rule.

CASE II.

167. Elimination by comparison.

1. Given $3x+5y=42$, and $2x+y=14$, to find the values of x and y.

OPERATION.

$$3x+5y=42 \ldots\ldots\ (1),$$
$$2x+\ y=14 \ldots\ldots\ (2).$$

From (1), by transposition, etc., $x=\dfrac{42-5y}{3}\ldots(3).$

" (2), " $x=\dfrac{14-y}{2}\ldots(4);$

therefore, by Ax. 7, $\dfrac{42-5y}{3}=\dfrac{14-y}{2};$

clearing of fractions, $84-10y=42-3y;$

whence, $7y=42,$

$y=\ 6.$

Substituting the value of y in (3), $x=\ 4.$

Hence, we have the following

RULE.—I. *Find the value of the same unknown quantity, in terms of the other, from each of the given equations.*

II. *Form an equation by placing these two values equal to each other.*

CASE III.

168. Elimination by substitution.

1. Given $3x+2y=16$, and $5x-3y=14$, to find the values of x and y.

OPERATION.

$$3x+2y=16 \ldots\ldots\ (1),$$
$$5x-3y=14 \ldots\ldots\ (2).$$

From (1), $y=\dfrac{16-3x}{2}\ldots(3).$

Substituting this value of y in (2),

$$5x - \frac{48 - 9x}{2} = 14;$$

clearing of fractions, $\quad 10x - 48 + 9x = 28;$

whence, $\quad x = 4.$

Substituting the value of x in (3), $\quad y = 2.$

Hence the following

RULE.—I. *Find the value of one of the unknown quantities, in terms of the other, from either of the given equations.*

II. *Substitute this value for the same unknown quantity in the other equation.*

CASE IV.

169. Elimination by indeterminate multipliers.

1. Given $2x + 3y = 23$, and $5x + 2y = 30$, to find the values of x and y.

If we multiply the first equation by a quantity, m, which is as yet undetermined, we have

$$2mx + 3my = 23m \ldots (1),$$

$$5x + 2y = 30 \quad \ldots (2).$$

Subtracting (2) from (1), and factoring with reference to x and y, we have

$$(2m - 5)x + (3m - 2)y = 23m - 30 \ldots (3).$$

It is evident that (3) is true, whatever be the value of m. We may therefore assume m to be of such a value that the coefficient of one of the unknown quantities shall become zero; this will eliminate that unknown quantity, by causing the term containing it to disappear. Thus, assume

$$2m - 5 = 0 \ldots (4);$$

whence, $\quad m = \frac{5}{2} \ldots (5).$

But if $2m - 5 = 0$, the first term of (3) is 0, and that equation becomes

$$(3m - 2)y = 23m - 30 \ldots (6);$$

whence, $\quad y = \frac{23m - 30}{3m - 2} \ldots (7).$

If we now substitute in (7) the value of m, as given in (5), we shall have

$$y=\frac{23\times\frac{5}{2}-30}{3\times\frac{5}{2}-2}=\frac{115-60}{15-4}=\frac{55}{11}=5\ldots(8).$$

In like manner we may eliminate x from (3). To accomplish this, assume

$$3m-2=0\ldots\ (9);$$

whence, $$m=\frac{2}{3}\ldots(10).$$

But if $3m-2=0$, (3) becomes

$$(2m-5)x=23m-30\ldots(11);$$

whence, $$x=\frac{23m-30}{2m-5}\ldots(12).$$

Substituting in (12) the value of m, as expressed in (10),

$$x=\frac{23\times\frac{2}{3}-30}{2\times\frac{2}{3}-5}=\frac{46-90}{4-15}=\frac{-44}{-11}=4\ldots(13).$$

This method of elimination is due to the French writer, Bezout. It is called the method of *indeterminate multipliers,* because the multipliers which we employ are at first *undetermined.* Strictly speaking, the multipliers thus used are not *indeterminate;* for, in order to effect the elimination of the unknown quantities, they must have certain definite values, which values are always determined in the course of the operation.

Recurring to the operation above, we notice that m has two values, thus:

$$m=\frac{5}{2}\ldots(1),$$

$$m=\frac{2}{3}\ldots(2).$$

The first value of m is the one by which x was eliminated; the second, the one by which y was eliminated. Resume the given equations,

$$2x+3y=23\ldots(1),$$
$$5x+2y=30\ldots(2).$$

It will be readily seen that if the first equation be multiplied by the first value of m, the coefficients of x will be alike in the two equations; and if the first equation be multiplied by the second value of m, the coefficients of y will be alike in the two equations. It is obvious, therefore, that the method of indeterminate multipliers is but a modification of the method of *addition and subtraction.*

Recurring again to the above operation, it is evident that if the *sum* instead of the *difference*, of equations (1) and (2) had been taken, the elimination might have been effected with equal facility. Hence,

RULE.—I. *Multiply one of the given equations by the indefinite factor* m, *and then take the sum or difference of this result and the other equation, factoring with reference to the unknown quantities.*

II. *Put the coefficient of one of the unknown quantities in this last equation equal to zero, and determine the value of* m; *then substitute this value in the equation containing* m, *and the result will be an equation of but one unknown quantity.*

170. In the reduction of simultaneous equations containing two unknown quantities, sometimes one of the preceding methods of elimination is preferable, and sometimes another, according to the special relations of the coefficients.

EXAMPLES FOR PRACTICE.

1. Given $\left\{\begin{array}{r} 8x + 5y = 68 \\ 12x + 7y = 100 \end{array}\right\}$ to find x and y.

Ans. $x = 6$; $y = 4$.

2. Given $\left\{\begin{array}{r} 5x + 2y = 19 \\ 7x - 6y = 9 \end{array}\right\}$ to find x and y.

Ans. $x = 3$; $y = 2$.

3. Given $\left\{\begin{array}{r} 3x + 7y = 79 \\ x + 4y = 38 \end{array}\right\}$ to find x and y.

Ans. $x = 10$; $y = 7$.

4. Given $\left\{\begin{array}{r} 5x - 3y = 36 \\ 2x + 9y = 96 \end{array}\right\}$ to find x and y.

Ans. $x = 12$; $y = 8$.

5. Given $\left\{\begin{array}{r} x + 17y = 54 \\ 3x - 25y = 10 \end{array}\right\}$ to find x and y.

Ans. $x = 20$; $y = 2$.

6. Given $\left\{\begin{array}{r} 5x - 4y = 40 \\ x - 5y = -97 \end{array}\right\}$ to find x and y.

Ans. $x = 28$; $y = 25$.

7. Given $\left\{\begin{array}{r} 8x + 15y = 9 \\ 6x - 12y = -1 \end{array}\right\}$ to find x and y.

Ans. $x = \frac{1}{2}$; $y = \frac{1}{3}$.

8. Given $\left\{\begin{array}{l} 7x + 7y = 30 \\ 3x + 4y = 17 \end{array}\right\}$ to find x and y.

Ans. $x = \frac{1}{7}$; $y = 4\frac{1}{7}$.

9. Given $\left\{\begin{array}{l} 8x + 3y = 25 \\ 5x - 6y = 55 \end{array}\right\}$ to find x and y.

Ans. $x = 5$; $y = -5$.

10. Given $\left\{\begin{array}{l} 15x - 8y = 9 \\ 10x + 4y = -43 \end{array}\right\}$ to find x and y.

Ans. $x = -2\frac{1}{5}$; $y = -5\frac{1}{4}$.

11. Given $\left\{\begin{array}{l} 9x - 5y = 950 \\ 2x - 3y = -450 \end{array}\right\}$ to find x and y.

Ans. $x = 300$; $y = 350$.

12. Given $\left\{\begin{array}{l} \frac{x}{2} - \frac{y}{4} = 20 \\ \frac{x}{12} + \frac{y}{8} = 10 \end{array}\right\}$ to find x and y.

Ans. $x = 60$; $y = 40$.

13. Given $\left\{\begin{array}{l} \frac{x}{2} + \frac{y}{3} = 8 \\ \frac{x}{3} - \frac{y}{5} = -1 \end{array}\right\}$ to find x and y.

Ans. $x = 6$; $y = 15$.

14. Given $\left\{\begin{array}{l} 3x - \frac{y}{2} = 3\frac{1}{2} \\ 4x - \frac{y}{5} = 7 \end{array}\right\}$ to find x and y.

Ans. $x = 2$; $y = 5$.

15. Given $\left\{\begin{array}{l} \frac{x}{8} + 8y = 194 \\ \frac{y}{8} + 8x = 131 \end{array}\right\}$ to find x and y.

Ans. $x = 16$; $y = 24$.

16. Given $\left\{\begin{array}{l} \frac{x}{3} + 3y = 21 \\ \frac{y}{3} + 3x = 29 \end{array}\right\}$ to find x and y.

Ans. $x = 9$; $y = 6$.

17. Given $\left\{\begin{array}{l} \frac{x}{7} + 7y = 99 \\ \frac{y}{7} + 7x = 51 \end{array}\right\}$ to find x and y.

Ans. $x = 7$; $y = 14$.

18. Given $\left\{\begin{array}{l} \frac{4}{x} - \frac{4}{y} = 1 \\ \frac{4}{x} - \frac{2}{y} = 1\frac{1}{2} \end{array}\right\}$ to find x and y.

Ans. $x = 2$; $y = 4$.

19. Given $\left\{\begin{array}{l} \frac{147}{x} - \frac{147}{y} = 28 \\ \frac{17}{x} + \frac{56}{y} = 13\frac{2}{3} \end{array}\right\}$ to find x and y.

Ans. $x = 3$; $y = 7$.

20. Given $\left\{\begin{array}{l} \frac{4x + 17}{4} = x + \frac{68}{x + y} \\ \frac{5y + 27}{5} = y + \frac{54}{x - y} \end{array}\right\}$ to find x and y.

Ans. $x = 13$; $y = 3$.

21. Given $\left\{\begin{array}{l} x - \frac{2y - x}{23 - x} = 20 - \frac{59 - 2x}{2} \\ y + \frac{y - 3}{x - 18} = 30 - \frac{73 - 3y}{3} \end{array}\right\}$ to find x and y.

Ans. $x = 21$; $y = 20$.

22. Given $\left\{\begin{array}{l} \frac{6x^2 - 24y^2 + 130}{2x - 4y + 3} = 3x + 6y + 1 \\ \frac{9xy - 110}{3x - 4} + \frac{151 - 16x}{4y - 1} = 3x \end{array}\right\}$ to find x and y.

Ans. $x = 9$; $y = 2$.

23. Given $\left\{\begin{array}{l} \frac{x + 3y}{3} - \frac{7x - 21}{6} = \frac{3x - 15}{4} - \frac{8x - 9y}{12} \\ \frac{2x + y}{2} - \frac{9x - 7}{8} = \frac{3y + 9}{4} - \frac{4x + 5y}{16} \end{array}\right\}$

to find x and y. *Ans.* $x = 9$; $y = 4$.

24. Given $\left\{\begin{aligned} ax + by &= d \\ a'x + b'y &= d' \end{aligned}\right\}$ to find x and y.

$$Ans.\ x = \frac{b'd - bd'}{b'a - ba'};\ y = \frac{a'd - ad'}{a'b - ab'}.$$

25. Given $\left\{\begin{aligned} \frac{x}{a} + \frac{y}{b} &= 2ab \\ \frac{x}{ab} + \frac{y}{ab} &= a + b \end{aligned}\right\}$ to find x and y.

$$Ans.\ x = a^2b;\ y = ab^2.$$

26. Given $\left\{\begin{aligned} ax + cy &= \frac{a^4 + c^4}{a^2c^2} \\ cx + ay &= \frac{a^2 + c^2}{ac} \end{aligned}\right\}$ to find x and y.

$$Ans.\ x = \frac{a}{c^2};\ y = \frac{c}{a^2}.$$

SIMPLE EQUATIONS

CONTAINING MORE THAN TWO UNKNOWN QUANTITIES.

171. If we have three or more simultaneous equations, they may be solved by successive eliminations, as follows:

$$\text{Given } \left\{\begin{aligned} 2x + 4y + 4z &= 18 \dots (1), \\ 3x + 3y + 2z &= 17 \dots (2), \\ 5x + 6y + 5z &= 32 \dots (3). \end{aligned}\right.$$

Multiplying (1) by 3,	$6x + 12y + 12z = 54 \dots (4);$
" (2) by 2,	$6x + 6y + 4z = 34 \dots (5);$
subtracting (5) from (4),	$6y + 8z = 20 \dots (6);$
multiplying (1) by 5,	$10x + 20y + 20z = 90 \dots (7);$
" (3) by 2,	$10x + 12y + 10z = 64 \dots (8);$
subtracting (8) from (7),	$8y + 10z = 26 \dots (9);$
multiplying (6) by 4,	$24y + 32z = 80 \dots (10);$
" (9) by 3,	$24y + 30z = 78 \dots (11);$
subtracting (11) from (10),	$2z = 2 \dots (12);$
whence,	$z = 1;$
substituting value of z in (9),	$y = 2;$
" values of y and z in (1),	$x = 3.$

INFERENCES.

1. If we have n equations, and proceed as above to combine one of them with each of the others, eliminating the same letter by each combination, we shall have $n-1$ derived equations containing all of the unknown quantities except the one eliminated.

2. If then we combine one of these derived equations with each of the others, eliminating another letter, we shall have $n-2$ derived equations, containing all of the unknown quantities except the two eliminated.

3. Since each succeeding group of derived equations consists of one less equation than the preceding group, it follows that if this process of successive elimination be continued, the $(n-1)$th group will consist of a single equation ; and this will contain all of the unknown quantities except the $n-1$ eliminated quantities. Hence,

4. If the number of original equations equals the number of unknown quantities, the final equation will contain but a single unknown quantity, the value of which may be found. By substituting this value in one of the equations of the preceding group, the value of a second unknown quantity may be determined ; and so on.

5. But if the number of original equations is less than the number of unknown quantities, the final equation will contain more than one unknown quantity, and will be indeterminate **(162)** ; consequently, the given equations will be indeterminate.

6. In the solution of two or more simultaneous equations of the first degree, by successive eliminations, the value of each letter is determined finally by a simple equation containing only that letter. And since every such equation can have only one root **(155)**, it follows that any group of simultaneous equations can be satisfied by only *one set of values* of the unknown quantities.

172. From the foregoing inferences we derive the following

RULE.—I. *Combine one of the given equations with each of the others, eliminating the same unknown quantity by each combination; then combine one of the new equations with each of the others, eliminating a second unknown quantity, and thus continue till a final equation is obtained, containing but one unknown quantity.*

II. *Solve this final equation, and find the value of the unknown quantity which it involves; substitute this value in an equation containing two unknown quantities, and thus find the value of a second; substitute these values in an equation containing three unknown quantities, and thus find the value of a third; and so on, till the values of all are found.*

PRACTICAL SUGGESTIONS.

This rule may be modified in certain cases, as follows:

1. Instead of combining the first equation with each of the others, we may pursue any order of combination, or adopt any one of the four methods of elimination, which seems best suited to the mutual relations of the coefficients. The following example will illustrate the precept just given:

$$\text{Given}\begin{cases} x + y + z = 9 \ldots (1), \\ x + 2y + 3z = 16 \ldots (2), \\ x + 3y + 4z = 21 \ldots (3). \end{cases}$$

Subtracting (1) from (2), $y + 2z = 7 \ldots (4)$;

" (2) from (3), $y + z = 5 \ldots (5)$;

" (5) from (4), $z = 2$;

substituting the value of z in (5), $y = 3$;

" values of y and z in (1), $x = 4$.

2. If two or more of the equations, taken together, do not contain all the unknown quantities, it is generally most convenient to employ these equations first, in the process of elimination. Thus,

$$\text{Given}\begin{cases} 2u - x + 3y + 2z = 19 \ldots (1), \\ 3u + 5x - 4y = 23 \ldots (2), \\ 4u + 3x = 32 \ldots (3), \\ 2u + 5x = 30 \ldots (4). \end{cases}$$

Multiplying (4) by 2, $4u + 10x = 60$;

bringing down (3), $4u + 3x = 32$;

by subtraction, $7x = 28$;

$x = 4$;

substituting in (3), $u = 5$;

" " (2), $y = 3$;

" " (1), $z = 2$.

3. When the coefficients sustain to each other relations of equality or symmetry, it is often convenient to employ an auxiliary quantity, as in the two examples which follow:

$$1.\ \text{Given}\ \begin{cases} u+v+x+y=14\ldots(1), \\ u+v+x+z=15\ldots(2), \\ u+v+y+z=16\ldots(3), \\ u+x+y+z=17\ldots(4), \\ v+x+y+z=18\ldots(5). \end{cases}$$

Since in each equation one letter is wanting, let

$$u+v+x+y+z=s;$$

then

$$\begin{aligned} s-z&=14, \\ s-y&=15, \\ s-x&=16, \\ s-v&=17, \\ s-u&=18. \end{aligned}$$

By addition, $5s-s=80$,

$$s=20.$$

Hence, by substituting the value of s in each equation, $\begin{cases} z=6, \\ y=5, \\ x=4, \\ v=3, \\ u=2. \end{cases}$

$$2.\ \text{Given}\ \begin{cases} x+4y+4z=300\ldots(1), \\ y+7x+7z=450\ldots(2), \\ z+9x+9y=490\ldots(3). \end{cases}$$

Assume	$x+y+z=s$;
equation (1) becomes	$4s-3x=300\ldots(4)$;
" (2) "	$7s-6y=450\ldots(5)$;
" (3) "	$9s-8z=490\ldots(6)$;
Multiplying (4) by 8,	$32s-24x=2400\ldots(7)$;
" (5) by 4,	$28s-24y=1800\ldots(8)$;
" (6) by 3,	$27s-24z=1470\ldots(9)$.
Adding (7), (8), and (9),	$87s-24s=5670$;
	$63s=5670$;
	$s=90$.
Substituting value of s in (4),	$x=20$;
" " " (5),	$y=30$;
" " " (6),	$z=40$.

EXAMPLES FOR PRACTICE.

Required the values of the unknown quantities in the following equations :

1. $\begin{cases} 2x + 4y - 3z = 22 \\ 4x - 2y + 5z = 18 \\ 6x + 7y - z = 63 \end{cases}$. *Ans.* $\begin{cases} x = 3, \\ y = 7, \\ z = 4. \end{cases}$

2. $\begin{cases} 3x + 9y + 8z = 41 \\ 5x + 4y - 2z = 20 \\ 11x + 7y - 6z = 37 \end{cases}$. *Ans.* $\begin{cases} x = 2, \\ y = 3, \\ z = 1. \end{cases}$

3. $\begin{cases} x + y + z = 31 \\ x + y - z = 25 \\ x - y - z = 9 \end{cases}$. *Ans.* $\begin{cases} x = 20, \\ y = 8, \\ z = 3. \end{cases}$

4. $\begin{cases} x + y + z = 26 \\ x - y = 4 \\ x - z = 6 \end{cases}$. *Ans.* $\begin{cases} x = 12, \\ y = 8, \\ z = 6. \end{cases}$

5. $\begin{cases} x - y - z = 6 \\ 3y - x - z = 12 \\ 7z - y - x = 24 \end{cases}$. *Ans.* $\begin{cases} x = 39, \\ y = 21, \\ z = 12. \end{cases}$

NOTE.—In the last example, assume $x + y + z = s$, and add this equation to each of the given equations. Then determine s as in Ex. 2, p. 115.

6. $\begin{cases} 2x = u + y + z \\ 3y = u + x + z \\ 4z = u + x + y \\ u = x - 14 \end{cases}$. *Ans.* $\begin{cases} u = 26, \\ x = 40, \\ y = 30, \\ z = 24. \end{cases}$

7. $\begin{cases} u + 3x - y - z = 7 \\ 2u - 2x + y + 3z = 8 \\ 3u - x + y - 4z = 8 \\ 4u + x - y - 2z = 7 \end{cases}$. *Ans.* $\begin{cases} u = 3, \\ x = 4, \\ y = 7, \\ z = 1. \end{cases}$

8. $\begin{cases} 5x - y + 7z = 61 \\ 4x + 3y + 3z = 8 \\ 3x - y - 5z = 3 \end{cases}$. *Ans.* $\begin{cases} x = 5, \\ y = -8, \\ z = 4. \end{cases}$

9. $\left\{\begin{array}{l} u+v+x+y+2z=52 \\ u+v+x+z+2y=50 \\ u+v+y+z+2x=48 \\ u+x+y+z+2v=46 \\ v+x+y+z+2u=44 \end{array}\right\}.$ *Ans.* $\left\{\begin{array}{l} u=4, \\ v=6, \\ x=8, \\ y=10, \\ z=12. \end{array}\right.$

10. $\left\{\begin{array}{r} 2x+y-2z=40 \\ 4y-x+3z=35 \\ 3u+t=13 \\ y+u+t=15 \\ 3x-y+3t-u=49 \end{array}\right\}.$ *Ans.* $\left\{\begin{array}{l} x=20, \\ y=10, \\ z=5, \\ u=4, \\ t=1. \end{array}\right.$

11. $\left\{\begin{array}{l} x+y-z=1 \\ 8x+3y-6z=1 \\ 3z-4x-y=1 \end{array}\right\}.$ *Ans.* $\left\{\begin{array}{l} x=2, \\ y=3, \\ z=4. \end{array}\right.$

12. $\left\{\begin{array}{l} 2u+2x+2y+z=-3 \\ 3u+3x+3z+2y=3 \\ 4u+4y+4z+3x=-2 \\ 5x+5y+5z+4u=2 \end{array}\right\}.$ *Ans.* $\left\{\begin{array}{l} u=-2, \\ x=2, \\ y=-3, \\ z=3. \end{array}\right.$

13. $\left\{\begin{array}{l} \frac{x}{2}+\frac{y}{3}+\frac{z}{4}=62 \\ \frac{x}{3}+\frac{y}{4}+\frac{z}{5}=47 \\ \frac{x}{4}+\frac{y}{5}+\frac{z}{6}=38 \end{array}\right\}.$ *Ans.* $\left\{\begin{array}{l} x=24, \\ y=60, \\ z=120. \end{array}\right.$

14. $\left\{\begin{array}{l} \frac{1}{x}+\frac{1}{y}=2 \\ \frac{1}{x}+\frac{1}{z}=3 \\ \frac{1}{y}+\frac{1}{z}=3 \end{array}\right\}.$ *Ans.* $\left\{\begin{array}{l} x=1, \\ y=1, \\ z=\frac{1}{2}. \end{array}\right.$

15. $\left\{\begin{array}{l} x+a=y+z \\ y+a=2x+2z \\ z+a=3x+3y \end{array}\right\}.$ *Ans.* $\left\{\begin{array}{l} x=\frac{1}{11}a, \\ y=\frac{5}{11}a, \\ z=\frac{7}{11}a. \end{array}\right.$

16. $\left\{\begin{array}{l} x+y+2z=2(b+c) \\ x+z+2y=2(a+c) \\ y+z+2x=2(a+b) \end{array}\right\}.$ *Ans.* $\left\{\begin{array}{l} x=a+b-c, \\ y=a+c-b, \\ z=b+c-a. \end{array}\right.$

17. $\left\{\begin{array}{l} 7x - 3y = a \\ 5y - 11x = a \\ 9y - 5z = a \end{array}\right\}$. *Ans.* $\left\{\begin{array}{l} x = 4a, \\ y = 9a, \\ z = 16a. \end{array}\right.$

18. $\left\{\begin{array}{l} \frac{x}{a} + \frac{y}{b} = \frac{x}{b} - \frac{y}{a} \\ x + y = \frac{4a^2b}{a^2-b^2} \end{array}\right\}$. *Ans.* $\left\{\begin{array}{l} x = \frac{2ab}{a-b}. \\ y = \frac{2ab}{a+b}. \end{array}\right.$

19. $\left\{\begin{array}{l} ax + by + cz = ab + ac + bc \\ a^2x + b^2y + c^2z = 3abc \\ \frac{x-c}{bc} + \frac{y-c}{ac} + \frac{y-a}{ac} + \frac{z-a}{ab} = 0 \end{array}\right\}$. *Ans.* $\left\{\begin{array}{l} x = \frac{bc}{a}. \\ y = \frac{ac}{b}. \\ z = \frac{ab}{c}. \end{array}\right.$

20. $\left\{\begin{array}{l} cx + y + az = 2a \\ c^2x + y + a^2z = 2ac \\ acx - y + acz = a^2 + c^2 \end{array}\right\}$. *Ans.* $\left\{\begin{array}{l} x = \frac{a+1}{c}, \\ y = a - c, \\ z = \frac{c-1}{a}. \end{array}\right.$

21. $\left\{\begin{array}{l} a^2x + ay + az = a \\ ax + a^2y + az = a^2 \\ ax + ay + a^3z = a^3 \end{array}\right\}$. *Ans.* $\left\{\begin{array}{l} x = -\frac{a+1}{a+2}, \\ y = \frac{1}{a+2}, \\ z = \frac{(a+1)^2}{a+2}. \end{array}\right.$

PROBLEMS

PRODUCING EQUATIONS CONTAINING TWO OR MORE UNKNOWN QUANTITIES.

173. Two or more equations are said to be *independent*, when they are not derived one from the other, and cannot be reduced to the same form; as

$$3x + y = 17,$$
$$2x + 3y = 23.$$

Equations derived from the same problem are independent, when they express different conditions of that problem.

174. We have seen that a group of equations will be determinate, when the number of equations is equal to the number of unknown quantities, but not otherwise (**172**, 4 and 5). Hence,

A problem will be capable of solution only when its conditions furnish as many independent equations as there are unknown symbols employed in the notation.

1. Find two numbers, such that twice the first plus three times the second is equal to 105; and three times the first plus twice the second is 95. *Ans.* First, 15; second, 25.

2. Find three numbers, such that the first with $\frac{1}{2}$ of the sum of the second and third shall be 120; the second with $\frac{1}{5}$ of the sum of the third and first shall be 90; and the sum of the three shall be 190. *Ans.* 50, 65, 75.

3. A sum of money was divided among three persons, A, B, and C, as follows: the share of A exceeded $\frac{4}{7}$ of the shares of B and C by $120; the share of B exceeded $\frac{3}{8}$ of the shares of A and C by $120; and the share of C exceeded $\frac{2}{9}$ of the shares of A and B by $120. What was each person's share?

Ans. A's, $600; B's, $480; C's, $360.

4. A and B, working together, can earn $40 in 6 days; A and C can earn $54 in 9 days; and B and C can earn $80 in 15 days. How much can each person earn in one day?

Ans. A, $$3\frac{2}{3}$; B, $3; C, $$2\frac{1}{3}$.

5. A man has 4 sons. The sum of the ages of the first, second, and third is 18 years; the sum of the ages of the first, second, and fourth is 16 years; the sum of the ages of the first, third, and fourth is 14 years; the sum of the ages of the second, third, and fourth is 12 years. What are their respective ages?

Ans. 8, 6, 4, and 2 years.

6. Three persons engaged in throwing dice, on certain conditions. In the first game, A forfeited to B and C, respectively, as many shillings as each of them had; in the second game, B forfeited to A and C, respectively, as many shillings as each of them then had; in the third game, C forfeited to A and B, respectively, as many shillings as each of them then had; they had then 16 shillings apiece. How many shillings had each at first?

Ans. A, 26; B, 14; C, 8.

7. A gentleman left a sum of money to be divided among four servants, so that the share of the first should be $\frac{1}{2}$ of the sum of the shares of the other three, the share of the second $\frac{1}{3}$ of the sum of the other three, and the share of the third $\frac{1}{4}$ of the sum of the other three. On making the division, the fourth had 14 dollars less than the first. Required the sum divided, and the several shares.

Ans. Sum divided, \$120; shares, \$40, \$30, \$24, and \$26.

8. A person has two horses and two saddles, the saddles being worth \$15 and \$10, respectively. Now the value of the better horse with the better saddle is $\frac{4}{3}$ of the value of the other horse and saddle; but the value of the better horse with the poorer saddle is $\frac{15}{13}$ of the value of the other horse and saddle. What are the values of the two horses? *Ans.* \$65 and \$50.

9. A vintner, in mixing sherry and brandy, finds that if he takes 2 parts of sherry to 1 of brandy, the mixture will be worth 78 shillings per dozen; but if he takes 7 parts of sherry to 2 of brandy, the mixture will be worth 79 shillings per dozen. What are the sherry and brandy worth per dozen?

Ans. Sherry, 81 shillings; brandy, 72 shillings.

10. Two persons, A and B, can perform a piece of work in 16 days. They work together for four days, when A is called off, and B is left to finish it, which he does in 36 days. In what time would each do it separately?

Ans. A, in 24 days; B, in 48 days.

11. What fraction is that, whose numerator being doubled, and denominator increased by 7, the value becomes $\frac{2}{3}$; but the denominator being doubled, and the numerator increased by 2, the value becomes $\frac{3}{5}$? *Ans.* $\frac{4}{5}$.

12. Two men were wishing to purchase a house together, valued at 240 dollars. Says A to B, "If you will lend me $\frac{2}{3}$ of your money I can purchase the house alone;" but says B to A, "If you will lend me $\frac{3}{4}$ of yours, I can purchase the house alone." How much money had each? *Ans.* A, \$160; B, \$120.

13. A pleasure party, having chartered a boat for a certain sum, found, on settling, that if their number had been 4 more, they would have had a shilling apiece less to pay; but if their

number had been 3 less, they would have had a shilling apiece more to pay. What was their number, and what had each to pay?

Ans. 24 persons; each paid 7 shillings.

14. A certain number consists of two places of figures, units and tens; the number is equal to 4 times the sum of its digits, and if 27 be added to the number, the order of the digits will be inverted. What is the number?

NOTE 1.—Let x represent the digit in the place of tens, and y the digit in place of units; then $10x + y$ will express the number.

Ans. 36.

15. A number is expressed by three figures whose sum is 11; the figure in the place of units is double that in the place of hundreds; and if 297 be added to the number, the result will be expressed by the same figures with their order reversed. What is the number? *Ans.* 326.

16. Divide the number 90 into three parts, such that twice the first part increased by 40, three times the second part increased by 20, and four times the third part increased by 10, may all be equal to one another.

Ans. First part, 35; second, 30; third, 25.

17. A person placed \$100000 out at interest, a part of it at 5 per cent., and the rest at 4 per cent.; the yearly interest received on the whole was \$4640. Required the two parts of the principal. *Ans.* \$64000 and \$36000.

18. A person put out a certain sum of money at interest at a certain rate. Another person put out \$10000 more than the first, at a rate per cent. greater by 1, and received an income greater by \$800. A third person put out \$15000 more than the first, at a rate per cent. greater by 2, and received an income greater by \$1500. Required the three principals, and the respective rates of interest.

NOTE 2.—To avoid the inconvenience of large numbers in the operation, put $a = 5000$; then $2a = 10000$, $3a = 15000$, $\frac{3a}{10} = 1500$, and $\frac{16a}{100} = 800$. In the final result, the value of a may be restored.

$$Ans. \begin{cases} \text{Principals, \$30000, \$40000, \$45000.} \\ \text{Rates, } \quad 4, \quad 5, \quad 6, \text{ per cent.} \end{cases}$$

19. If B's age be subtracted from A's, the difference will be C's age; if 5 times B's age and twice C's age be added together, and from their sum A's age be subtracted, the remainder will be 147; and the sum of the three ages is 96. Required the ages of A, B, and C, respectively. *Ans.* A's, 48; B's, 33; C's, 15.

20. Find what each of three persons, A, B, and C, is worth, knowing, 1st, that what A is worth added to 3 times what B and C are worth, is equal to 4700 dollars; 2d, that what B is worth added to 4 times what A and C are worth, is equal to 5800 dollars; 3d, that what C is worth added to 5 times what A and B are worth, is equal to 6300 dollars. *Ans.* A, \$500; B, \$600; C, \$800.

21. A grocer sold 50 pounds of tea at an advance of 10 per cent. on the cost, and 30 pounds of coffee at an advance of 20 per cent. on the cost, and received for the whole \$27.40, gaining \$2.90. What was the cost per pound of the tea and coffee?

Ans. Tea, \$.40; coffee, \$.15.

22. Five persons, A, B, C, D, E, play at cards; after A has won one half of B's money, B one-third of C's, C one-fourth of D's, D one-sixth of E's, they have each \$30. How much had each to begin with? *Ans.* A, \$11; B, \$38; C, \$33; D, \$32; E, \$36.

23. Three brothers desired to make a purchase, requiring \$2000 of each. The first wanted, in addition to his own money, $\frac{1}{2}$ of the money of the second; the second wanted, in addition to his own, $\frac{1}{3}$ of the money of the third; and the third wanted, in addition to his own, $\frac{1}{4}$ of the money of the first. How much money had each? *Ans.* 1st, \$1280; 2d, \$1440; 3d, \$1680.

24. A courier was sent from A to B, a distance of 147 miles; after 28 hours had elapsed, a second courier was sent from the same place, who overtook the first just as he entered B. Now the time required by the first to travel 17 miles, added to the time required by the second to travel 56 miles, is $13\frac{2}{3}$ hours. How many miles did each travel per hour? *Ans.* 1st, 3 miles; 2d, 7 miles.

25. Find two numbers, such that if $\frac{1}{2}$ of the greater be added to $\frac{1}{3}$ of the less, the sum shall be 13; and if $\frac{1}{2}$ of the less be subtracted from $\frac{1}{3}$ of the greater, the remainder will be nothing.

Ans. 18 and 12.

26. Find three numbers of such magnitudes, that the first added to $\frac{1}{2}$ of the sum of the other two, the second added to $\frac{1}{3}$ of the sum of the other two, and the third added to $\frac{1}{4}$ of the sum of the other two, may each be equal to 51. *Ans.* 15, 33, and 39.

27. Said A to B and C, "If each of you will give me 4 sheep, I shall have 4 more than both of you will have left." Said B to A and C, "If each of you will give me 4 sheep, I shall have twice as many as both of you will have left." Said C to A and B, "If each of you will give me 4 sheep, I shall have three times as many as both of you will have left." How many sheep had each?

Ans. A, 6; B, 8; C, 10.

28. What fraction is that, to the numerator of which if 1 be added, the fraction will be $\frac{1}{3}$; but if to the denominator 1 be added, the fraction will be $\frac{1}{4}$? *Ans.* $\frac{4}{15}$.

29. What fraction is that, to the numerator of which if 2 be added, the fraction will be $\frac{5}{7}$; but if to the denominator 2 be added, the fraction will be $\frac{1}{3}$? *Ans.* $\frac{3}{7}$.

30. Four persons, A, B, C, D, were engaged together in mowing for 4 successive days. The first day A worked 1 hour, B 3 hours, C 2 hours, and D 2 hours, and all together mowed 1 acre; the second day A worked 3 hours, B 2 hours, C 4 hours, and D 11 hours, and all together mowed 2 acres; the third day A worked 5 hours, B 4 hours, C 12 hours, and D 5 hours, and all together mowed 3 acres; the fourth day A worked 9 hours, B 7 hours, C 6 hours, and D 8 hours, and all together mowed 4 acres. How many hours would each alone require to mow 1 acre?

Ans. A, 5 hours; B, 6 hours; C, 12 hours; D, 15 hours.

31. If A give B \$5 of his money, B will have twice as much money as A has left; and if B give A \$5, A will have thrice as much as B has left. How much has each?

Ans. A, \$13; B, \$11.

32. A corn factor mixes wheat flour, which cost him 10 shillings per bushel, with barley flour, which cost 4 shillings per bushel, in such a ratio as to gain $43\frac{3}{4}$ per cent. by selling the mixture at 11 shillings per bushel. Required the ratio.

Ans. The ratio is 14 bushels of wheat flour to 9 of barley.

33. There is a number consisting of two digits, which number divided by 5 gives a certain quotient and a remainder of 1, and the same number divided by 8 gives another quotient and a remainder of 1. Now the quotient obtained by dividing by 5 is twice the value of the digit in the tens' place, and the quotient obtained by dividing by 8 is equal to 5 times the digit in the units' place. What is the number? *Ans.* 41.

34. The four classes in a certain college are to compete for four prizes, amounting in the aggregate to $119, and the prize money is to be raised by contribution, on the following conditions, namely: that the members of the class whose candidate obtains the 1st prize shall each pay one dollar, and the class whose candidate obtains the 2d prize shall pay the remainder. Now it is found that if a senior gets the 1st prize and a junior the 2d, each junior will pay $\frac{1}{2}$ of a dollar; if a junior gets the 1st prize and a sophomore the 2d, each sophomore will pay $\frac{1}{3}$ of a dollar; if a sophomore gets the 1st prize and a freshman the 2d, each freshman will pay $\frac{1}{4}$ of a dollar; and if a freshman gets the 1st prize and a senior the 2d, each senior will pay $\frac{1}{5}$ of a dollar. Of how many members does each class consist?

Ans. { Freshman, 104; Sophomore, 93; Junior, 88; Senior, 75.

35. Find four numbers, such that if 3 times the first be added to the second, 4 times the second be added to the third, 5 times the third be added to the fourth, and 6 times the fourth be added to the first, each sum shall be 359. *Ans.* 95, 74, 63, 44.

GENERAL SOLUTION OF PROBLEMS.

175. In the preceding problems, the given quantities have been expressed by numbers, and it has been required simply to determine the values of the unknown quantities from the numerical relations thus expressed.

If, however, the given quantities in any problem be represented by *letters*, the solution will give rise to a *formula*, showing not only the value of the unknown quantity, but indicating the precise operations to be performed in order to obtain this value. This is called a *general solution* of the problem.

176. When any particular problem has been proposed, we may, by simply varying the numbers, form other problems of the same kind or class; and the solutions of all the problems of the class will require exactly the same operations. Hence,

177. The ***General Solution*** of a problem is the process of obtaining a formula which shall express, in known terms, the values of the unknown quantities in the given problem, or in any problem of its class.

178. An ***Arbitrary Quantity*** is one to which any value may be assigned at pleasure, in a general formula or equation.

179. For illustration, let the following questions be proposed:

1. What number is that whose third part exceeds its fourth part by 6?

Instead of confining our attention to the particular numbers here given, we may first investigate the problem under a general form, as follows:

What number is that whose mth *part exceeds its* nth *part by* a?

Let x represent the number; then by the conditions,

$$\frac{x}{m} - \frac{x}{n} = a \ldots (1);$$

clearing of fractions, $nx - mx = amn \ldots (2)$;

whence, $$x = \frac{amn}{n - m} \ldots (3).$$

Equation (3) is the formula which indicates the operations to be performed in solving all questions of this class.

If in this formula we put $m=3$, $n=4$, and $a=6$, we shall have

$$x = \frac{6 \times 3 \times 4}{4 - 3} = 72,$$

the number required by the particular question as at first proposed.

2. What number is that whose fifth part exceeds its seventh part by 12?

To obtain the number by the formula, let $m = 5$, $n = 7$, and $a = 12$; then

$$x = \frac{12 \times 5 \times 7}{7 - 5} = 210, \text{ } Ans.$$

EXAMPLES FOR PRACTICE.

1. Divide the number n into two such parts that the greater increased by a shall be equal to the less increased by b.

Ans. Greater, $\dfrac{n+b-a}{2}$; less, $\dfrac{n+a-b}{2}$.

2. In the last example, what will be the two parts if $n = 84$, $a = 16$, and $b = 58$? *Ans.* 63 and 21.

3. The sum of three numbers is s; the second exceeds the first by a, and the third exceeds the second by b. Required the numbers.

Ans. $\dfrac{s-2a-b}{3}$, $\dfrac{s+a-b}{3}$, $\dfrac{s+a+2b}{3}$.

4. My indebtedness to three persons, A, B, and C, amounts to a dollars; and I owe B, n times the sum which I owe A, and C, m times the sum which I owe A. What is my indebtedness to A?

Ans. $\$\dfrac{a}{1+n+m}$.

5. In the last example, what is the sum due to A when $a = \$786$, $n = 2$, and $m = 3$? *Ans.* \$131.

6. A person engaged to work a days on these conditions: For each day he worked he was to receive b cents, and for each day he was idle he was to forfeit c cents; at the end of a days he received d cents. How many days was he idle?

Ans. $\dfrac{ab-d}{b+c}$ days.

7. My horse and saddle are together worth a dollars, and my horse is worth n times the price of my saddle. What is the value of each?

Ans. Saddle, $\$\dfrac{a}{n+1}$; horse, $\$\dfrac{na}{n+1}$.

8. The rent of an estate is n per cent. greater this year than it was last. This year it is a dollars; what was it last year?

Ans. $\dfrac{100a}{100+n}$ dollars.

9. A person after spending a dollars more than $\frac{1}{7}$ of his income, had remaining b dollars more than $\frac{1}{3}$ of it. Required his income.

Ans. $\dfrac{21(a+b)}{11}$ dollars.

10. A person after spending a dollars more than $\frac{1}{m}$th of his income, had remaining b dollars more than $\frac{1}{n}$th of it. Required his income.

Ans. $\dfrac{mn(a+b)}{mn-m-n}$ dollars.

11. If A can perform a certain piece of work in a days, and B can do the same in b days, and C the same in c days, in how many days can all together perform the work?

Ans. $\dfrac{abc}{ab+ac+bc}$ days.

12. In the last example, what will be the time required, when $a=6$, $b=8$, and $c=12$? *Ans.* $2\frac{2}{3}$ days.

13. If from a times a certain number, c be subtracted, the remainder will be equal to b times the number increased by d. Required the number.

Ans. $\dfrac{c+d}{a-b}$.

14. A farmer would mix oats worth a cents a bushel with peas worth b cents a bushel, to form a mixture of c bushels worth d cents a bushel. How many bushels of each kind must he take?

Ans. Oats, $\dfrac{c(d-b)}{a-b}$; peas, $\dfrac{c(a-d)}{a-b}$.

15. There were a boys in one party, and b boys in another, and each party had the same number of nuts. Each boy in the first party snatched m nuts from the second party, and ate them; then each boy in the second party snatched m nuts from the first party, and ate them. Each party then divided the nuts remaining to it equally among its members, when the boys in the two parties found that they had the same number of nuts apiece; how many nuts had each party at first? *Ans.* $m(a+b)$.

16. Find four numbers, such that if a times the first be added to the second, b times the second be added to the third, c times the third be added to the fourth, and d times the fourth be added to the first, each sum shall be m.

$$\textit{Ans.}\ \begin{cases} \text{1st, } \dfrac{m(bcd-cd+d-1}{abcd-1}; \text{ 2d, } \dfrac{m(acd-ad+a-1)}{abcd-1}; \\[2ex] \text{3d, } \dfrac{m(abd-ab+b-1)}{abcd-1}; \text{ 4th, } \dfrac{m(abc-bc+c-1)}{abcd-1}. \end{cases}$$

17. A sent n pupils regularly to a certain school during a term of a days, and B sent m pupils regularly to another school for a term of b days. The two schools had the same number of pupils in attendance, and raised the same amount of money by rate-bill. There were c days' absence allowed for at the school to which A sent, and d days' absence at the school to which B sent; and A and B found that they had equal sums to pay. What was the number of pupils attending each school?

Ans. $\dfrac{bcm - adn}{ab(m - n)}$.

18. Divide the number m into four parts, such that the second shall be a times the first, the third a times the second, and the fourth a times the third.

Ans. 1st part, $\dfrac{m}{a^3 + a^2 + a + 1}$.

19. The sum of two numbers is s, and their difference is d. Required the numbers.

Ans. Greater, $\dfrac{s + d}{2}$; less, $\dfrac{s - d}{2}$.

20. There are three numbers, such that the sum of the first and second is a, the sum of the first and third is b, and the sum of the second and third is c. What are the numbers?

Ans. 1st, $\dfrac{a + b - c}{2}$; 2d, $\dfrac{a + c - b}{2}$; 3d, $\dfrac{b + c - a}{2}$.

21. There is a number consisting of two digits; the number is equal to a times the sum of its digits; and if c be added to the number, the order of the digits will be reversed. Required the two digits.

Ans. Digit in units' place, $\dfrac{c(10 - a)}{9(11 - 2a)}$.
Digit in tens' place, $\dfrac{c(a - 1)}{9(11 - 2a)}$.

22. Find what each of three persons, A, B, and C, is worth, knowing, 1st, that what A is worth added to l times what B and C are worth is equal to p; 2d, that what B is worth added to m times what A and C are worth is equal to q; 3d, that what C is worth added to n times what A and B are worth is equal to r.

We give here a solution of this example, partly to illustrate the method of simplifying algebraic formulas by the use of auxiliary quantities.

Let $x =$ A's money, $y =$ B's money, $z =$ C's money.

Then, by the conditions, $\begin{cases} x + ly + lz = p \dots\dots\dots (1), \\ y + mx + mz = q \dots\dots\dots (2), \\ z + nx + ny = r \dots\dots\dots (3). \end{cases}$

Assume, $$x + y + z = s \dots\dots\dots (4).$$

Multiplying (4) by l, m, and n, successively, and subtracting (1) from the first product, (2) from the second, and (3) from the third, and reducing the respective remainders, we have

$$\begin{cases} x = \dfrac{ls - p}{l - 1} \dots (5), \\ y = \dfrac{ms - q}{m - 1} \dots (6), \\ z = \dfrac{ns - r}{n - 1} \dots (7). \end{cases}$$

Adding (5), (6), and (7), we obtain

$$s = \frac{ls - p}{l - 1} + \frac{ms - q}{m - 1} + \frac{ns - r}{n - 1} \dots\dots\dots (8), \text{ or}$$

$$s = \left(\frac{l}{l-1} + \frac{m}{m-1} + \frac{n}{n-1}\right)s - \left(\frac{p}{l-1} + \frac{q}{m-1} + \frac{r}{n-1}\right) \dots (9).$$

Now the parenthetical expressions in equation (9) are known quantities. Hence, to simplify the results,

Put $$\begin{cases} a = \dfrac{l}{l - 1} + \dfrac{m}{m - 1} + \dfrac{n}{n - 1} \dots\dots\dots (10), \\ b = \dfrac{p}{l - 1} + \dfrac{q}{m - 1} + \dfrac{r}{n - 1} \dots\dots\dots (11). \end{cases}$$

Equation (9) then becomes $$s = as - b \dots\dots\dots (12);$$

whence, $$s = \frac{b}{a - 1} \dots\dots\dots (13).$$

Substituting the value of s in (5), $x = \dfrac{lb - p(a - 1)}{(l - 1)(a - 1)}$;

" " " (6), $y = \dfrac{mb - q(a - 1)}{(m - 1)(a - 1)}$;

" " " (7), $z = \dfrac{nb - r(a - 1)}{(n - 1)(a - 1)}$.

DISCUSSION OF PROBLEMS INVOLVING SIMPLE EQUATIONS.

180. The ***Discussion*** of a problem consists in attributing certain values and relations to the arbitrary quantities which enter the equation, and in interpreting the results.

181. When a problem has been solved in a general manner, we may proceed to make an unlimited number of suppositions upon the arbitrary quantities involved in the formulas, and thus obtain a variety of results. But our experience with algebraic equations would lead us to expect that the problem might not be rational, or possible, under every hypothesis. Now the principal object in the discussion of a problem is to examine the peculiar or anomalous forms which present themselves, and ascertain whether the problem is rational or absurd, or how it is to be understood, under the suppositions which lead to these peculiarities. We shall commence with the

INTERPRETATION OF NEGATIVE RESULTS.

1. What number must be added to a that the sum may be b?

Let x represent the required number. Then, by the conditions of the question,

$$a + x = b \; \ldots\ldots (1);$$

whence,

$$x = b - a \ldots (2).$$

This is a general solution, a and b being arbitrary quantities.

First, suppose $a = 20$ and $b = 28$; then by the formula,

$$x = 28 - 20 = 8,$$

a result which satisfies the conditions; for, we perceive that 8 is the number which must be added to 20, or a, to make 28, or b. Second, suppose $a = 20$ and $b = 12$; then by the formula,

$$x = 12 - 20 = -8,$$

a negative result.

In order to ascertain the meaning of the minus sign in this case, let us enunciate the question according to the supposition that gave this result; thus,

What number must be added to 20, that the sum may be 12?

Now as 20 is greater than 12, no number can be added to 20, arithmetically, to make 12. The problem is therefore impossible under the second hypothesis, if understood in an arithmetical sense.

We shall find, however, that if we change the words *added to*, and *sum*, to their opposites, the result will be a rational question, of which 8, the absolute value of x, is the answer. Thus,

What number must be *subtracted from* 20, that the *difference* may be 12? *Ans.* 8.

We observe, moreover, that the negative result, -8, will satisfy the *equation* of the problem, under the second hypothesis. Thus,

$$20 + (-8) = 12;$$

or,

$$20 - 8 = 12.$$

That is, 12 is really the *algebraic sum* of 20 and -8.

2. A man dying left two sons, the elder of whom was a years of age, and the younger b years of age. In how many years after the death of the father was the elder son twice as old as the younger son?

Let x represent the number of years; then by the conditions,

$$a + x = 2(b + x) \ldots (1);$$

whence,

$$x = a - 2b \ldots. (2).$$

Since a and b are arbitrary quantities, suppose $a = 30$ and $b = 12$. Then by the formula,

$$x = 30 - 24 = 6.$$

This result will satisfy the conditions arithmetically; for, if the elder son was 30, and the younger son 12 years old, at the death of the father, then in 6 years the age of the elder was $30 + 6 = 36$ years, and the age of the younger was $12 + 6 = 18$ years.

Again, suppose $a = 30$ and $b = 18$. Then by the formula,

$$x = 30 - 36 = -6.$$

To interpret the negative result in this case, we observe that the problem under the second hypothesis is impossible, if understood in the exact sense of the enunciation. For, when the elder son was 30 and the younger son 18 years old, the younger son was already more than one half as old as the elder; and as their ages are equally increased by any lapse of time, it is evident that

the elder son could never become twice as old as the younger son, *after* the death of the father. Let us therefore modify the general problem as follows:

A man dying left two sons, the elder of which was a years of age, and the younger b years of age. How many years *before* the death of the father was the elder son twice as old as the younger?

If we let x represent the number of years, then the solution will be as follows:

$$a - x = 2(b - x) \ldots (1);$$

whence, $$x = 2b - a \ldots\ldots (2).$$

Now suppose, as before, that $a = 30$ and $b = 18$. Then by the new formula,

$$x = 36 - 30 = 6,$$

a result which will satisfy the modified conditions; for, six years *before* the death of the father, the age of the elder was $30 - 6 = 24$, and the age of the younger was $18 - 6 = 12$.

From the foregoing discussions we draw the following inferences:

1. *When the solution of a problem by a simple equation gives a negative result, the minus sign indicates that the problem is impossible, if understood in the exact sense of the enunciation.*

2. *The impossibility thus indicated consists in adding a quantity when it should be subtracted; or in treating a quantity as reckoned or applied in a certain direction, when it should be reckoned or applied in an opposite direction.*

3. *In all such cases, an analogous problem may be formed, involving no impossibility, by changing the terms of the absurd condition to their opposites; and the answer to the new question will be found by simply changing the sign of the negative result already obtained.*

182. The foregoing discussions give a more extensive signification to the plus and minus signs, and lead to a more general view of positive and negative quantities, than was presented in a former section.

Let us recur to the problem of the two sons. In the solution of this problem, we employ the signs, + and —, in the *state-*

ment, merely to indicate addition and subtraction. But in the *result,* these signs have a very different use; they enable us to distinguish the circumstances or conditions of the quantities which they affect. Thus, under the first hypothesis, the period of time represented by x occurred after the death of the father, and in the result is found to be affected by the plus sign; but under the second hypothesis, the period represented by x occurred before the death of the father, and in the result is found to be affected by the minus sign.

Thus we perceive that plus and minus, in Algebra, are not *symbols of operation* merely, but also *symbols of relation,* serving to distinguish quantities in opposite conditions or circumstances.

It should be observed, however, that this enlarged use of the plus and minus signs is not entirely conventional or arbitrary, but is necessarily involved in the more extended signification given to the terms *addition* and *subtraction,* in Algebra. Indeed, we shall never meet with a negative result in the solution of problems, so long as the language conforms, in the exact arithmetical sense, to the facts of the case.

EXAMPLES FOR PRACTICE.

1. What number is that whose fourth part exceeds its third part by 12? *Ans.* -144.

The question is impossible, if understood in an arithmetical sense. Let the pupil modify the enunciation, and solve the new problem.

2. A man when he was married was 30 years old, and his wife 15. How many years must elapse before his age will be three times the age of his wife? *Ans.* $-7\frac{1}{2}$ years.

That is, their ages bore the specified relation $7\frac{1}{2}$ years *before,* not after, their marriage.

3. The sum of two numbers is s, and their difference d; what are the numbers? *Ans.* Greater, $\frac{s}{2}+\frac{d}{2}$; less, $\frac{s}{2}-\frac{d}{2}$.

How shall the result be interpreted when $s=120$ and $d=160$?

4. Two men, A and B, commenced trade at the same time, A having 3 times as much money as B. When A had gained $400

and B \$150, A had twice as much money as B; how much did each have at first? *Ans.* A was in debt \$300, and B \$100.

5. A man worked 7 days, and had his son with him 3 days, and received for wages 22 shillings, and the board of his son and himself while at work. He afterward worked 5 days, and had his son with him one day, and received 18 shillings. What were his daily wages, and what the daily wages of his son?

Ans. The father received 4 shillings per day, and paid 2 shillings for his son's board.

6. A man worked for a person 10 days, having his wife with him 8 days, and his son 6 days, and received \$10.30 as compensation for all three; at another time he worked 12 days, his wife 10 days, and son 4 days, and received \$13.20; at another time he worked 15 days, his wife 10 days, and his son 12 days, at the same rates as before, and received \$13.85. What were the daily wages of each?

Ans. He received \$.75 for himself, \$.50 for his wife, and *paid* \$.20 for his son's board.

7. A man worked 10 days for his neighbor, his wife 4 days, and son 3 days, and received \$11.50; at another time he served 9 days, his wife 8 days, and his son 6 days, at the same rates as before, and received \$12.00; a third time he served 7 days, his wife 6 days, and his son 4 days, at the same rates as before, and he received \$9.00. What were the daily wages of each?

Ans. Husband's wages, \$1.00; wife's, 0; son's, \$.50.

8. What fraction is that which becomes $\frac{3}{5}$ when 1 is added to its numerator, and $\frac{4}{7}$ when 1 is added to its denominator?

Ans. In an arithmetical sense, there is no such fraction. The algebraic expression, $\frac{-10}{-15}$ will give the required results.

How shall the enunciation be modified, to form an analogous question involving no absurdity?

9. Four merchants, A, B, C, D, find by their balance sheets that if they unite in a firm, receiving the assets and assuming the liabilities of each, they will have a joint net capital of \$5780. If A, B, and C unite on the same conditions, their joint capital will be \$7950; if B, C, and D unite, their joint capital will be \$2220;

and if C, D, and A unite, their joint capital will be $7320. Required the net capital or the net insolvency of each.

10. Two men were traveling on the same road towards Boston, A at the rate of a miles per hour, and B at the rate of b miles per hour. At 6 o'clock A was at a point m miles from Boston, and at 10 o'clock B was at a point n miles from Boston. Find the time when A passed B upon the road.

Ans. $\frac{m-n-4b}{a-b}$ hours after 6 o'clock.

11. What time of day will be indicated by the preceding formula, if $m=36$, $n=28$, $a=5$, and $b=3$? *Ans.* 4 o'clock.

12. There are two numbers whose difference is a; and if 3 times the greater be added to 5 times the less, the sum will be b. What are the numbers? *Ans.* Greater, $\frac{b+5a}{8}$; less, $\frac{b-3a}{8}$.

How shall this result be interpreted if $a=24$ and $b=48$?

NOTHING AND INFINITY.

183. The limits between which all absolute values are comprised are *nothing* and *infinity;* and the symbols by which these limits are denoted are 0 and ∞.

184. In certain algebraic investigations it is convenient to employ these symbols in connection with each other and the ordinary symbols of quantity. They may thus sustain the relations of divisor, dividend, quotient, or factor. Such relations, however, cannot really exist except between symbols of *quantity.* Hence, in Algebra, 0 does not always signify merely *absence of value;* nor does ∞ represent *infinity, in the highest sense of the word.*

The more complete definition of these symbols may be given as follows:

185. The symbol 0, called *nothing,* or *zero,* may be used to denote the absence of value, or to represent a quantity less than any assignable value.

186. The symbol ∞, called *infinity,* is used to represent a quantity greater than any assignable value.

INTERPRETATION OF THE FORMS $\frac{A}{0}$, $\frac{A}{\infty}$, $\frac{0}{A}$, AND $\frac{0}{0}$.

187. In order to understand the signification of the expressions

$$\frac{A}{0},\ \frac{A}{\infty},\ \frac{0}{A},\ \text{and}\ \frac{0}{0},$$

we may consider the symbols 0 and ∞ as resulting from an arbitrary or varying quantity, made to diminish until it becomes indefinitely small, or to increase until it becomes indefinitely great.

188. Let $\frac{a}{b}$ represent a fraction, a and b being arbitrary quantities. And let it be remembered that the value of a fraction depends simply upon the *relative* values of the numerator and denominator.

1. If the denominator b is made to diminish, becoming less and less continually, while the numerator a remains unchanged, the value of the fraction must increase, becoming greater and greater continually (**119**, II); and thus when the denominator b becomes less than any assignable quantity, or 0, the value of the fraction must become greater than any assignable quantity, or ∞. Hence, we conclude that

$$\frac{a}{0} = \infty. \qquad \text{That is,}$$

A finite quantity divided by zero is an expression for infinity.

2. If the denominator b is made to increase, becoming greater and greater continually, while the numerator a remains unchanged, the value of the fraction must diminish, becoming less and less continually (**119**, II); and when the denominator b becomes greater than any assignable quantity, or ∞, the value of the fraction must become less than any assignable quantity, or 0. Hence,

$$\frac{a}{\infty} = 0. \qquad \text{That is,}$$

A finite quantity divided by infinity is an expression for zero or nothing.

3. If the numerator a is made to diminish, becoming less and less continually, while the denominator b remains unchanged, the

value of the fraction must diminish continually (**119**, I); and when a becomes less than any assignable quantity, or 0, the value of the fraction also must become 0. Hence,

$$\frac{0}{b} = 0.$$

That is,

Zero divided by a finite quantity is an expression for nothing or zero.

4. If both a and b are made to diminish simultaneously, but in such a manner as to preserve their *relative value*, then the value of the fraction will remain unchanged, however small the terms become (**119**, III); and when both a and b become less than any assignable quantity, or 0, we shall have the expression $\frac{0}{0}$ representing the value of $\frac{a}{b}$. And since this value may be any quantity whatever, we conclude that $\frac{0}{0}$ represents an indeterminate quantity. That is,

Zero divided by zero is a symbol of indetermination.

NOTE.—If it should be difficult for any one to conceive how both terms of a fraction may, by being diminished, become nothing at the same time, and yet preserve the same relative value to the last, it may be useful to consider the following illustrations:

Take the fraction $\frac{c}{d}$, in which d represents the diameter of a circle, and c the circumference. Now the diameter and circumference of a circle have the same ratio to each other, whatever the dimensions of the circle. Hence, if the circle be made to diminish until it shall become a point, or vanish, both terms of the fraction, $\frac{c}{d}$, will diminish, and become 0 at the same instant, *the value of the fraction remaining the same throughout*, and reducing to the form, $\frac{0}{0}$, at the instant the circle vanishes. Now the ratio of the diameter to the circumference of a circle is known to be 3.1416 — ; hence, in the present case, we shall have

$$\frac{0}{0} = 3.1416.$$

Again, let s represent the side of a square and d the diagonal. Then we have the well-known ratio

$$\frac{d}{s} = \sqrt{2}.$$

If the square is supposed to diminish by insensible degrees, both d and s will vanish at the same instant, and we shall have finally

$$\frac{0}{0} = \sqrt{2}.$$

PROBLEM OF THE COURIERS.

189. The anomalous forms which have been explained in the last article will now be viewed in connection with a general problem, involving certain relations of *motion*, *time*, and *distance*. The discussion will also confirm our interpretation of negative results.

PROBLEM.—Two couriers, A and B, were traveling along the same road and in the same direction, namely, from C′ toward C; the former going at the rate of a miles per hour, and the latter at the rate of b miles per hour. At 12 o'clock, A was at a certain point P, and B was d miles in advance of A, in the direction of C. It is required to find *when* and *where* the couriers were together.

This problem is entirely general, and we do not know from the enunciation whether the couriers were together after, or before 12 o'clock; nor whether the place of meeting was to the right, or to the left of P. But in order to effect a statement of the problem, we will suppose the required time to be *after* 12 o'clock. Then we must regard time after 12 o'clock as positive, and time before 12 o'clock as negative; also, distance reckoned from P toward C as positive, and distance reckoned from P toward C′ as negative. Accordingly,

Let $t =$ the number of hours after 12 o'clock;
$x =$ the distance from P to the point of meeting.

And since A traveled at the rate of a miles per hour, and B at the rate of b miles per hour, we have

$x = at =$ distance traveled by A after 12 o'clock;
$bt =$ " " " B " " ".

But since A and B were d miles apart, at 12 o'clock, we have

$$at - bt = d,$$

$$t = \frac{d}{a-b} \ldots (1);$$

$$x = \frac{ad}{a-b} \ldots (2).$$

We may now discuss this problem with reference to the time t, and the distance x, which are the two unknown elements.

I. Suppose $a > b$.

Under this hypothesis the values of both t and x will be positive, because the common denominator, $a - b$, is positive. Now since t is positive, we conclude that the two couriers came together after 12 o'clock; and as x is positive, we infer that the point of meeting is somewhere to the right of P.

These conclusions agree with each other, and are consistent with the conditions of the problem. For, the supposition that a is greater than b implies that A was traveling faster than B. A would therefore gain upon B, and overtake him sometime after 12 o'clock, and at a point situated in the direction of C.

II. Suppose $a < b$.

Then in equations (1) and (2) the denominator, $a - b$, is negative, and consequently both t and x will be negative.

This implies that t and x must be taken in a sense contrary to that in which they were employed under the hypothesis (I), where they were positive; that is, the time when the couriers were together was *before* 12 o'clock, and the place of meeting was situated to the left of P.

This interpretation, also, agrees with the conditions of the problem, under the present hypothesis. For, if a is less than b, then B was traveling faster than A; and as B was in advance of A at 12 o'clock, he must have passed A before that time, somewhere to the left of P, in the direction of C'.

III. Suppose $a = b$.

Under this hypothesis we shall have $a - b = 0$, and

$$t = \frac{d}{0} = \infty, \quad \text{and} \quad x = \frac{ad}{0} = \infty.$$

Now, according to these results, t, the time to elapse before the couriers are together, is greater than any assignable quantity, or infinity; therefore they can never be together. And likewise x, the distance from P to the supposed point of meeting, is greater than any assignable quantity, or infinity; hence there can be no such point, however distant from P.

This interpretation is in accordance with the conditions of the problem, under the present hypothesis. For, at 12 o'clock the two

couriers were d miles apart; and if $a = b$, they were traveling at equal rates, neither approaching nor separating. Hence, they could always continue in motion, and remove to any distance from P, without meeting.

IV. Suppose $d = 0$, and $a > b$ or $a < b$.

Then we shall have

$$t = \frac{0}{a - b} = 0, \text{ and } x = \frac{0}{a - b} = 0.$$

That is, both the time and distance are *nothing*. These results must be interpreted to mean that the couriers were together at 12 o'clock, at the point P, and at no other time or place.

And this interpretation is also confirmed by the conditions of the problem. For, if $d = 0$, then at 12 o'clock B must have been with A, at the point P. And if $a > b$ or $a < b$, the couriers were traveling at different rates, and must be either approaching or receding from each other at all times except at the moment of passing; hence, they could be together only at a single point.

V. Suppose $d = 0$, and $a = b$.

We shall then have

$$t = \frac{0}{0}, \text{ and } x = \frac{0}{0}.$$

Here the values of both t and x are represented by the symbol of indetermination, which signifies that the time and the distance may be anything whatever; and we infer that the couriers must be together at all times, and at any distance from P.

And this conclusion is evidently confirmed by the conditions of the problem. For, if $d = 0$, the couriers were together at 12 o'clock; and if $a = b$, they were traveling at equal rates, and would never separate.

190. To the foregoing interpretations, there is an apparent exception in the case of the expression $\frac{0}{0}$. For, a fraction which is not indeterminate will reduce to this form, if its terms contain a common factor that becomes zero under the hypothesis.

Thus, in the solution of a problem, suppose

$$x = \frac{a^3 - b^3}{a^2 - b^2} \dots (1).$$

If we put $a = b$, which implies that $a - b = 0$, then

$$x = \frac{0}{0},$$

and the value of x appears to be indeterminate. Let us, however, cancel the common factor, $a - b$, from both terms of the fraction in equation (1); we shall obtain,

$$x = \frac{a^2 + ab + b^2}{a + b} \ldots (2).$$

If in this reduced equation, we make $a = b$, as before, we shall have a determinate value for x. Thus,

$$x = \frac{3a}{2}.$$

Hence the following practical direction:

In the discussion of a problem, a fractional result should be reduced to its lowest terms before making the hypothesis.

191. We are sometimes liable to an error in the reduction of an equation, in consequence of a false assumption respecting the value of an expression reducible to the form of indetermination.

1. Let us take the equation,

$$\frac{6x + 7}{x + 2} = \frac{6x - 12}{x - 2} \ldots\ldots (1).$$

Reducing second member, $\quad \frac{6x + 7}{x + 2} = 6 \ldots\ldots\ldots\ldots (2);$

clearing of fractions, $\quad 6x + 7 = 6x + 12 \ldots\ldots (3);$

transposing and factoring, $\quad (6 - 6)\,x = 5 \ldots\ldots\ldots\ldots (4);$

whence, $\quad x = \frac{5}{6 - 6} = \frac{5}{0} \ldots (5),$

or (**188**, 1), $\quad x = \infty.$

This result is erroneous. To obtain the true root of equation (1), multiply both members by $(x + 2)(x - 2)$; we shall obtain,

$$6x^2 - 5x - 14 = 6x^2 - 24;$$

whence, $\quad 5x = 10,$

or, $\quad x = 2.$

Now we observe that if this true value of x be substituted in the second member of equation (1), it will reduce to the form $\frac{0}{0}$;

and the mistake in our first solution was made in assuming that $\frac{6x-12}{x-2}=6$, a conclusion which would be correct in all cases except when $x=2$.

2. If we make two assumptions that are inconsistent, respecting the values of quantities reducible to the form of indetermination, the result will be an algebraic absurdity.

Thus, take the identical equation,

$$8+20=8+20\ldots\ldots(1).$$

By transposition, $$8-8=20-20\ldots\ldots(2);$$

dividing by $4-4$, $$\frac{8-8}{4-4}=\frac{20-20}{4-4}\ldots\ldots(3);$$

factoring, $$\frac{2(4-4)}{4-4}=\frac{5(4-4)}{4-4}\ldots(4);$$

suppressing common factor, $$2=5\ \ldots\ldots\ldots(5).$$

Equation (5) is absurd. But this equation is not correctly derived from (3) or (4). In (3), both numerators and both denominators are zero. Hence (3) may be written,

$$\frac{0}{0}=\frac{0}{0},$$

a result which involves no absurdity, and certainly gives no authority for saying that 2 is equal to 5.

193. To afford the pupil further exercise in the interpretation of anomalous forms, we give the following

EXAMPLES.

1. A cistern has four pipes communicating with it. If all be opened together, and left running for 15 hours, the cistern will be filled; but if the first run only 5 hours, the second 8 hours, the third 7 hours, and the fourth 3 hours, the cistern will be but one half full; if the first run 3 hours, the second 4 hours, the third 3 hours, and the fourth 1 hour, only $\frac{1}{5}$ of the cistern will be filled; and if the first run 4 hours, the second 2 hours, the third 3 hours, and the fourth 2 hours, only $\frac{1}{4}$ of the cistern will be filled. In what time would the cistern be filled by each pipe alone?

2. A can earn 5 dollars, and B 3 dollars, per day. In 2 days A will have a certain sum, and in 4 days B will have 2 dollars more than this sum. How many days hence will A and B have the same sum?

3. An astronomer being asked the period of a comet's revolution, answered, that if from 3 times the period 10 years be subtracted, and to 4 times the period 8 years be added, the former result would be equal to $\frac{3}{4}$ of the latter.* Required the period.

4. Two teachers, A and B, have the same monthly wages. A is employed 9 months in the year, and his annual expenses are \$450; B is employed 6 months in the year, and his annual expenses are \$300. Now A lays up in two years as much as B does in 3 years. Required the monthly wages of each.

FORMULA FOR TIME APPLIED TO CIRCULAR MOTION.

194. The Problem of the Couriers gave us the formula,

$$t = \frac{d}{a - b},$$

in which a and b are the respective rates of motion, d the distance to be gained, and t the time to elapse before the couriers will be together.

But the relations of these quantities will not be changed, if we suppose the path of motion to be a curve, instead of a straight line. The above formula will therefore apply to the hands of a clock moving around the dial-plate, or to the planets moving in the circle of the heavens. It will thus afford a direct solution to the following problems:

1. *The hour and minute hands of a clock are together at* 12 *o'clock; when are they next together?*

The circumference of the dial-plate is divided into 12 spaces. The minute hand moves over these 12 spaces while the hour hand moves over one of them; and when the minute hand has gained upon the hour hand a whole circumference, the two hands will be together.

Taking one of these spaces for the unit of distance, and one hour for the unit of time, we have

$$a = 12, \quad b = 1, \quad \text{and} \quad d = 12,$$

to substitute in the formula. Hence,

$$t = \frac{12}{12-1} = \frac{12}{11} = 1\text{ h. } 5\text{ m. } 27\tfrac{3}{11}\text{ s.}$$

2. *At what time between 2 and 3 o'clock will the hour and minute hands of a clock be together?*

In this case, the minute hand must evidently gain *two revolutions*, or 24 spaces. Hence, $d=24$; and we have by the formula,

$$t = \frac{24}{11} = 2\text{ h. } 10\text{ m. } 54\tfrac{6}{11}\text{ s.}$$

3. *What time between 2 and 3 o'clock will the hour and minute hands be at right angles to each other?*

In this case the minute hand must gain $2\frac{1}{4}$ revolutions; that is, $d = 12 \times 2\frac{1}{4} = 27$. Hence,

$$t = \frac{27}{11} = 2\text{ h. } 27\text{ m. } 16\tfrac{4}{11}\text{ s.}$$

4. *What time between 5 and 6 o'clock will the two hands of a clock be in the same straight line?*

Here the minute hand must gain $5\frac{1}{2}$ revolutions; and $d=12 \times 5\frac{1}{2} = 66$. Hence,

$$t = \frac{66}{11} = 6\text{ h.}$$

That is, the hands make a right line at 6 o'clock, a result manifestly true.

We will now apply this formula to certain motions of the heavenly bodies. It is known that the moon has a real motion around the earth from west to east. The sun also has an *apparent* motion in the same direction, in consequence of the *real* motion of the earth around the sun. The time of *new moon* is when the moon is in the direction of the sun from the earth, or when the moon is passing the sun, in her motion. With this explanation we present the following problem:

5. *The average daily motion of the moon around the circle of the heavens is* 13.1764°, *and the apparent daily motion of the sun in the same direction is* .98565°. *Required the time from one new moon to another.*

To apply the formula, we have $d = 360°$, $a = 13.1764°$. $b = .98565°$, and $a - b = 12.19075°$. Hence,

$$t = \frac{360}{12.19075} = 29\text{ d. } 12\text{ h. } 44\text{ m. } 33\text{ s.}$$

6. *The planet Venus, as seen from the sun, describes an arc of* 1° 36′ *per day, and the earth, as seen from the same point, describes an arc of* 59′. *At what intervals of time will these two bodies come in a line with the sun and on the same side of it?*

Here $d = 360° = 21600'$, $a = 1°\ 36'$, and $b = 59'$. Hence, $a - b = 37'$, and we have,

$$t = \frac{21600}{37} = 583.8 \text{ days, nearly.}$$

The data in the last example were not taken with extreme accuracy, the object being mainly to illustrate a method. More exact data would have given 583.92 days.

INEQUALITIES.

195. An ***Inequality*** is an expression signifying that one quantity is greater, or less, than another; as

$$a > b, \quad \text{and} \quad c < d.$$

In every inequality, the part on the left of the sign is the *first member*, and the part on the right the *second member*.

196. In treating of inequalities, the terms *greater* and *less*, must be understood in their algebraic sense, which may be defined as follows:

Of any two quantities, as a *and* b, a *is the greater when* a − b *is positive, and* a *is the less when* a − b *is negative.*

197. From this definition it follows, that

Any negative quantity is less than zero; and of two negative quantities, the greater is the one which has the less number of units.

Thus, $-2 < 0$, because $-2 - 0 = -2$, a negative result; and $-3 > -5$, because $-3 - (-5) = +2$, a positive result.

198. Two inequalities are said to ***subsist in the same sense,*** when the first member is the greater in both, or the less in both. Thus $a > d$ and $c > d$, or $u < z$ and $x < y$, are inequalities which subsist in the same sense. But the inequalities, $m > n$ and $p < q$, subsist in a contrary sense.

PROPERTIES OF INEQUALITIES.

199. Inequalities are frequently employed in mathematical investigations; and to facilitate their use, it is necessary to establish the following properties:

I. *An inequality will continue in the same sense, if the same quantity be added to, or subtracted from, each member.*

For, suppose

$$a > b.$$

Then according to **196**, $a - b$ is positive. Hence,

$$(a \pm c) - (b \pm c)$$

is positive, and consequently,

$$a \pm c > b \pm c.$$

It follows obviously from the principle just established,

1. That a term may be transposed from one member of an inequality to another, by changing its sign.

2. That if an equation be added to an inequality, member to member, or subtracted from it in like manner, the result will be an inequality subsisting in the same sense.

II. *If an inequality be subtracted from an equation, member from member, the sign of inequality will be reversed.*

For, suppose

$$x = y \quad \text{and} \quad a > b;$$

then we shall have

$$(x - a) - (y - b) = b - a,$$

a negative quantity (**196**); hence,

$$x - a < y - b.$$

III. *If the signs of all the terms of an inequality be changed, the sign of inequality will be reversed.*

For to change the signs of all the terms is equivalent to subtracting each member from $0 = 0$.

IV. *If two or more inequalities subsisting in the same sense, be added, member to member, the resulting inequality will subsist in the same sense as the given inequalities.*

For if $\quad a > b, \quad a' > b', \quad a'' > b'', \quad \ldots\ldots,$

then from **196**,

$$a - b, \quad a' - b', \quad a'' - b'', \quad \ldots\ldots,$$

are all positive; and the sum of these quantities,

$$a - b + a' - b' + a'' - b'', \text{ or } (a + a' + a'') - (b + b' + b''),$$

is therefore positive. Hence,

$$a + a' + a'' > b + b' + b''.$$

It is evident that if one inequality be subtracted from another established in the same sense, the result will not always be an inequality subsisting in the same sense. Thus, it is evident that we may have

$$a > b \quad \text{and} \quad a' > b',$$

in which $a - a'$ may be greater than $b - b'$, less than $b - b'$, or equal to $b - b'$.

V. *If one inequality be subtracted from another subsisting in a contrary sense, the result will be an inequality subsisting in the same sense as the minuend.*

For, if $\quad a > b \ldots (1),$

and $\quad a' < b' \ldots (2);$

then $a - b$ is positive and $a' - b'$ is negative; therefore, $a - b - (a' - b')$, or its equal $(a - a') - (b - b')$ must be positive, and we shall have

$$a - a' > b - b',$$

an inequality subsisting in the same sense as (1).

If (1) be subtracted from (2), member from member, it can be shown, in like manner, that

$$a' - a < b' - b.$$

VI. *An inequality will still subsist in the same sense, if both members be multiplied or divided by the same positive quantity.*

For suppose m to be essentially positive, and

$$a > b.$$

Then since $a - b$ is positive, we shall have both $m(a - b)$ and $\frac{1}{m}(a - b)$ positive. Therefore,

$$ma > mb \quad \text{and} \quad \frac{a}{m} > \frac{b}{m}.$$

VII. *If both members of an inequality be multiplied or divided by the same negative quantity, the sign of inequality will be reversed.*

For, to multiply or divide by a negative quantity will change the signs of all the terms, and consequently reverse the sign of inequality (III).

VIII. *If two inequalities subsisting in the same sense be multiplied together, member by member, the sign of inequality remains the same when more than two of the members are positive, but is reversed when more than two of the members are negative.*

That is,

Multiply	$a > b,$	$-a > -b,$	$-a > -b,$	$a > b.$
By	$a' > b',$	$-a' > -b',$	$a' > -b',$	$a' > -b'.$
Products,	$aa' > bb',$	$aa' < bb',$	$-aa' < bb',$	$aa' > -bb'.$

The first two results are evident from the fact that when the two members of an inequality are both positive, the greater member has the greatest numerical value; but when the two members are both negative, the greater member has the least numerical value.

The other two results are evident from the fact that any positive quantity is greater than any negative quantity.

It will be found that if two of the four members are positive and two negative, the result will be indefinite.

SOLUTION OF INEQUALITIES.

200. The ***Solution*** of an inequality consists in transforming it in such a manner that one member shall be the unknown quantity standing alone, and the other member a known expression. The inequality will then denote one *limit* of the unknown quantity.

201. The principles just established may now be applied in the solution of inequalities of the first degree.

Thus, let it be required to find the limit of x in the inequality,

$$\frac{x}{2}+\frac{2x}{5}>\frac{3x}{4}+\frac{9}{4}.$$

Multiplying both sides by 20,

$$10x+8x>15x+45;$$

transposing and collecting terms,

$$3x>45;$$

dividing by 3,

$$x>15.$$

EXAMPLES FOR PRACTICE.

1. $5x>\frac{3x}{2}+14.$ *Ans.* $x>4.$

2. $\frac{2x}{5}-\frac{2x}{3}>\frac{2x}{5}-2.$ *Ans.* $x<3.$

3. $\frac{5x}{8}+\frac{5}{4}<\frac{11}{6}+\frac{7x}{12}.$ *Ans.* $x<14.$

4. $\frac{3x}{4}-\frac{x-1}{2}<6x-\frac{20x+13}{4}.$ *Ans.* $x>5.$

5. $ax-b>cx+d.$ *Ans.* $x>\frac{b+d}{a-c}.$

6. $\frac{x-a}{b}<1-\frac{x}{a}.$ *Ans.* $x<a.$

7. $(a-x)(m-x)-a(m-c)<x^2-\frac{a^2c}{m}.$ *Ans.* $x>\frac{ac}{m}.$

202. If there be given an inequality and an equation, containing two unknown quantities, the limit of each unknown quantity may be found, by a process of elimination.

1. Given $2x+5y>16$ and $2x+y=12$, to find the limits of x and y.

If we subtract the equation from the inequality, the result will be an inequality subsisting in the same sense (**199**, I, 2), and x will be eliminated. Thus,

From $\qquad 2x + 5y > 16 \ldots (1),$

subtract $\qquad 2x + y = 12 \ldots (2);$

$$4y > 4;$$

$$y > 1.$$

If we substitute 1 for y in the equation, the first member will be made less than the second; and we shall have

$$2x + 1 < 12,$$

whence, $\qquad x < 5\tfrac{1}{2}.$

The limit of x may be found in a different manner, as follows:

From equation (2), $\qquad y = 12 - 2x.$

Substituting this value of y in (1), we have

$$2x + 60 - 10x > 16,$$

whence, $\qquad -8x > -44,$

or, $\qquad x < 5\tfrac{1}{2}.$

Thus we may eliminate between equalities and inequalities, either by addition and subtraction, or by substitution. Let it be remembered, however, that when an inequality is *subtracted* from an equation, the sign of inequality will be reversed (**199**, II).

EXAMPLES FOR PRACTICE.

1. Given $2x + 4y > 30$ and $3x + 2y = 31$, to find the limits of x and y. *Ans.* $x < 8$; $y > 3\tfrac{1}{2}$.

2. Given $4x - 3y < 15$ and $8x + 2y = 46$, to find the limits of x and y. *Ans.* $x < 5\tfrac{1}{4}$; $y > 2$.

3. Given $7x - 10y < 59$ and $4x + 5y = 68$, to find the limits of x and y. *Ans.* $x < 13$; $y > 3\tfrac{1}{5}$.

4. Given $5x + 3y > 121$ and $7x + 4y = 168$, to find the limits of x and y. *Ans.* $x < 20$; $y > 7$.

5. Given $\dfrac{x-4}{8} - \dfrac{y-10}{6} > 1$ and $\dfrac{3x-24}{4} + \dfrac{x-y}{2} = 13$, to find the limits of x and y. *Ans.* $x < 22\tfrac{2}{7}$; $y < 17\tfrac{5}{7}$.

SECTION III.

POWERS AND ROOTS.

INVOLUTION.

203. A ***Power*** of a quantity is the product of factors each of which is equal to that quantity. A quantity is said to be *raised* or *involved* when any power of it is found.

204. ***Involution*** is the process of raising a quantity to any given power.

205. Involution is indicated by an exponent, which expresses the name of the power, and shows how many times the quantity is taken as a factor.

Thus, let a represent any quantity; then,

The *first* power of a is		$a = a^1$;
" *second* "	"	$aa = a^2$;
" *third* "	"	$aaa = a^3$;
" *fourth* "	"	$aaaa = a^4$;
" n^{th} "	"	$aaa\ldots = a^n$;

206. The ***Square*** of a quantity is its second power; and the ***Cube*** of a quantity is its third power.

207. A ***Perfect Power*** is a quantity that can be exactly produced by taking some other quantity a certain number of times as a factor. Thus, $x^2 - 2xy + y^2$ is a perfect power, because it is equal to $(x - y)(x - y)$.

POWERS OF MONOMIALS.

208. A simple factor may be raised to any power by giving it an exponent which expresses the name or degree of the required power. And if a quantity consists of two or more factors, it is evident that as often as the quantity is repeated, each factor will be repeated. Thus,

$$(ab)^2 = ab \times ab = aa \times bb = a^2b^2.$$

And in general, if $abc \ldots . k$ represent the product of any number of factors, and n any exponent, we shall have

$$(abc \ldots . k)^n = a^n b^n c^n \ldots . k^n. \qquad \text{That is,}$$

The n^{th} power of the product of two or more factors is equal to the product of the n^{th} powers of those factors.

209. If it be required to involve a quantity which is already a power, the exponent of the quantity will be taken as many times as there are units in the exponent of the required power.

Thus,

$$(a^m)^2 = a^m \times a^m \qquad = a^{m+m} \quad = a^{2m};$$
$$(a^m)^3 = a^m \times a^m \times a^m = a^{m+m+m} = a^{3m}.$$

And in general, a^m raised to the n^{th} power will be

$$(a^m)^n = a^{mn}. \qquad \text{That is,}$$

If the m^{th} power of a quantity be raised to the n^{th} power, the result will be a power of the quantity expressed by the product of m and n.

210. With respect to signs, it is obvious that if a positive quantity be involved to any power, the result will be positive.

But if a negative quantity be involved, the successive powers will be alternately positive and negative; for, it has been shown that the product of an even number of negative factors is positive, and the product of an odd number of negative factors is negative (**67**).

To deduce this law of signs in an experimental way, let it be required to involve $-a$ to successive powers. By the principles of multiplication, we shall have

$$(-a)^2 = (-a) \times (-a) = +a^2;$$
$$(-a)^3 = (+a^2) \times (-a) = -a^3;$$
$$(-a)^4 = (-a^3) \times (-a) = +a^4;$$
$$(-a)^5 = (+a^4) \times (-a) = -a^5.$$

And in general,

$$(-a)^n = \pm a^n,$$

the plus sign in the second member being used when n is *even*, and the minus sign when n is *odd*. Hence,

1. *All powers of a positive quantity are positive.*

2. *The odd powers of a negative quantity are negative, but the even powers are positive.*

211. From the foregoing principles relating to the involution of a monomial, we derive the following

RULE.—I. *Raise the numeral coefficients to the required power.*

II. *Multiply the exponent of each letter by the exponent of the required power.*

III. *When the quantity involved is negative, give the odd powers the minus sign.*

EXAMPLES FOR PRACTICE.

1. Raise x^3 to the 4th power. *Ans.* x^{12}.
2. Raise y^7 to the 3d power. *Ans.* y^{21}.
3. Raise x^n to the 6th power. *Ans.* x^{6n}.
4. Raise x^m to the n^{th} power. *Ans.* x^{mn}.
5. Raise ax^2 to the 3d power. *Ans.* a^3x^6.
6. Raise ab^2x^4 to the 2d power. *Ans.* $a^2b^4x^8$.
7. Raise $5a^2x$ to the 3d power. *Ans.* $125a^6x^3$.
8. Raise $8a^2b^3$ to the 2d power. *Ans.* $64a^4b^6$.
9. Raise $-4a$ to the 4th power. *Ans.* $256a^4$.
10. Raise $-4a$ to the 3d power. *Ans.* $-64a^3$.
11. Required the 7th power of $-a^2x^3$. *Ans.* $-a^{14}x^{21}$.
12. Required the 4th power of $-3cd^2$. *Ans.* $81c^4d^8$.

Find the values of the following indicated powers:

13. $(6ab^2)^3$. *Ans.* $216a^3b^6$.
14. $(-5a^3b^4)^3$. *Ans.* $-125a^9b^{12}$.
15. $(a^mb^n)^4$. *Ans.* $a^{4m}b^{4n}$.
16. $(-a^m)^2$. *Ans.* a^{2m}.
17. $(-x^n)^5$. *Ans.* $-x^{5n}$.
18. $(-3a^cb^c)^2$. *Ans.* $9a^{2c}b^{2c}$.
19. $(-2a^mx^{2n})^7$. *Ans.* $-128a^{7m}x^{14n}$.
20. $(-abc)^m$. *Ans.* $\pm a^mb^mc^m$.

212. If it be required to raise a^m to the m^{th} power, we shall have

$$(a^m)^m = a^{m\times m} = a^{m^2},$$

an expression which denotes that power of a whose exponent is m^2. If we put $m = 3$, then $a^{m^2} = a^9$.

Expressions like the above may frequently occur in algebraic operations.

EXAMPLES.

Find the value of each of the following expressions:

1. $(x^m y^n)^n$. *Ans.* $x^{mn}y^{n^2}$.
2. $(x^m y^n)^m$. *Ans.* $x^{m^2}y^{mn}$.
3. $\left(x^{m^2}\right)^m$. *Ans.* x^{m^3}.
4. $\left(x^{m^2}\right)^{m^2}$. *Ans.* x^{m^4}.
5. $(x^{m-1}y)^{m+1}$. *Ans.* $x^{m^2-1}y^{m+1}$.
6. $\left(ab^n c^{n^2} d^{n^3}\right)^n$. *Ans.* $a^n b^{n^2} c^{n^3} d^{n^4}$.

POWERS OF FRACTIONS.

213. If a fraction be raised to any power, both numerator and denominator will be raised to the same power.

1. Required the 3d power of $\frac{a}{c}$.

$$\left(\frac{a}{c}\right)^3 = \frac{a}{c} \times \frac{a}{c} \times \frac{a}{c} = \frac{a \times a \times a}{c \times c \times c} = \frac{a^3}{c^3}, \textit{ Ans.}$$

Hence, to raise a fraction to any power, we have the following

RULE.—*Raise both numerator and denominator to the required power.*

EXAMPLES FOR PRACTICE.

1. Involve $\frac{3a}{bc^2}$ to the 2d power. *Ans.* $\frac{9a^2}{b^2c^4}$.
2. Involve $\frac{a^2}{3x^4}$ to the 3d power. *Ans.* $\frac{a^6}{27x^{12}}$.

3. Involve $\frac{4a^2b}{7x}$ to the 5th power. *Ans.* $-\frac{1024a^{10}b^5}{16807x^5}$.

4. Involve $-\frac{ax^3}{xy}$ to the 4th power. *Ans.* $\frac{a^4x^{12}}{x^4y^4}$.

5. Raise $\frac{5}{abc}$ to the 6th power. *Ans.* $\frac{15625}{a^6b^6c^6}$.

6. Raise $\frac{2x^m}{3a^nb^c}$ to the 5th power. *Ans.* $\frac{32x^{5m}}{243a^{5n}b^{5c}}$.

7. Raise $-\frac{abc}{xyz}$ to the n^{th} power. *Ans.* $\pm\frac{a^nb^nc^n}{x^ny^nz^n}$.

8. Find $\left(\frac{a^n}{c^m}\right)^{mn}$. *Ans.* $\frac{a^{mn^2}}{c^{m^2n}}$.

9. Find $\left(\frac{x^s}{y^2}\right)^s$. *Ans.* $\frac{x^{s^2}}{y^{2s}}$.

DISCUSSION OF NEGATIVE INDICES.

214. It has been shown in previous articles that

$$a^m \times a^n = a^{m+n},\ \frac{a^m}{a^n} = a^{m-n},\ \text{and}\ (a^m)^n = a^{mn},$$

where m and n are *positive whole numbers.* It remains to be shown that the above relations hold true when one or both of the exponents are *negative.* And in this investigation it is sufficient to remember that a quantity with a negative exponent is equal to the reciprocal of the same quantity with a positive exponent **(88,** 2).

I. To prove that $a^m \times a^n = a^{m+n}$ universally, m and n being integers.

1. Suppose *one* of the exponents to be negative; or let

$$n = -n'.$$

Then $$a^m \times a^n = a^m \times a^{-n'} = \frac{a^m}{a^{n'}} = a^{m-n'} = a^{m+n}.$$

2. Suppose *both* exponents are negative; or let

$$m = -m' \quad \text{and} \quad n = -n'.$$

Then

$$a^m \times a^n = a^{-m'} \times a^{-n'} = \frac{1}{a^{m'}} \times \frac{1}{a^{n'}} = \frac{1}{a^{m'+n'}} = a^{-m'-n'} = a^{m+n}.$$

II. To prove that $\frac{a^m}{a^n} = a^{m-n}$ universally, m and n being integers.

1. Suppose the exponent of the numerator to be negative; or let
$$m = -m'.$$

Then $$\frac{a^m}{a^n} = \frac{a^{-m'}}{a^n} = \frac{1}{a^{m'+n}} = a^{-m'-n} = a^{m-n}.$$

2. Suppose the exponent of the denominator to be negative; or let
$$n = -n'.$$

Then $$\frac{a^m}{a^n} = \frac{a^m}{a^{-n'}} = a^m \times a^{n'} = a^{m+n'} = a^{m-n}.$$

3. Suppose both exponents are negative; or let
$$m = -m' \quad \text{and} \quad n = -n'.$$

Then $$\frac{a^m}{a^n} = \frac{a^{-m'}}{a^{-n'}} = \frac{a^{n'}}{a^{m'}} = a^{n'-m'} = a^{m-n}.$$

III. To prove that $(a^m)^n = a^{mn}$ universally, m and n being integers.

1. Suppose n to be negative; or let
$$n = -n'.$$

Then $$(a^m)^n = (a^m)^{-n'} = \frac{1}{(a^m)^{n'}} = \frac{1}{a^{mn'}} = a^{-mn'} = a^{mn}.$$

2. Suppose m to be negative; or let
$$m = -m'.$$

Then $$(a^m)^n = (a^{-m'})^n = \left(\frac{1}{a^{m'}}\right)^n = \frac{1}{a^{m'n}} = a^{-m'n} = a^{mn}.$$

3. Suppose both m and n to be negative; or let
$$m = -m' \quad \text{and} \quad n = -n'.$$

Then $$(a^m)^n = (a^{-m'})^{-n'} = \left(\frac{1}{a^{m'}}\right)^{-n'} = \left(\frac{a^{m'}}{1}\right)^{n'} = a^{m'n'} = a^{mn}.$$

Hence, in all algebraic operations, the same rules will apply to negative exponents as to positive. That is, if two powers of the same quantity be given, then the exponent of their product will be equal to the *algebraic sum* of the given exponents, and the exponent of their quotient will be equal to the *algebraic difference* of the given exponents.

EXAMPLES.

215. Find the value of each of the following expressions:

1. $(a^{-2}b)^3$. *Ans.* $a^{-6}b^3$.
2. $(b^{-3}c^2)^{-2}$. *Ans.* b^6c^{-4}.
3. $(2x^3y^{-m})^{-3}$. *Ans.* $\frac{1}{8}x^{-9}y^{3m}$.
4. $(4a^mb^{-n})^2$. *Ans.* $16a^{2m}b^{-2n}$.
5. $(-c^2d^{-x}m^4)^5$. *Ans.* $-c^{10}d^{-5x}m^{20}$.
6. $(3a^{-2}xy^{-1})^{-4}$. *Ans.* $\frac{1}{81}a^8x^{-4}y^4$.
7. $(-a^my^{-n})^m$. *Ans.* $\pm a^{m^2}y^{-mn}$.
8. $\left(x^{-m^2}\right)^{m^{-3}}$. *Ans.* $x^{-m^{-1}}$.
9. $(4a^3x^{-2})^2 \times (a^{-5}x^4)$. *Ans.* $16a$.
10. $(a^{2m}b^{-3m})^2 \times (a^{-2m}b^{-m})^{-2}$. *Ans.* $a^{8m}b^{-4m}$.

POWERS OF POLYNOMIALS.

216. A polynomial may be raised to any power by actual multiplication. Thus, if the quantity be multiplied by itself, the product will be the second power; if the second power be multiplied by the quantity, the product will be the third power; and so on. Hence the following

RULE.—*Multiply the quantity by itself in continued multiplication, till it has been taken as many times as a factor as there are units in the exponent of the required power.*

NOTE.—It may be well to observe that in involution we may often reach the same result by different processes. Thus, we have $a^6=a^5\times a=a^4\times a^2=(a^3)^2=(a^2)^3$.

EXAMPLES FOR PRACTICE.

Expand the following expressions:

1. $(2x^2+3y)^2$. *Ans.* $4x^4+12x^2y+9y^2$.
2. $(5x-y^2)^3$. *Ans.* $125x^3-75x^2y^2+15xy^4-y^6$.
3. $(1+2x-3x^2)^2$. *Ans.* $1+4x-2x^2-12x^3+9x^4$.

4. $(3a + 2b + c)^3$.

Ans. $27a^3 + 54a^2b + 27a^2c + 36ab^2 + 36abc + 8b^3 + 9ac^2 + 12b^2c + 6bc^2 + c^3$.

5. $(a + b)^7$.

Ans. $a^7 + 7a^6b + 21a^5b^2 + 35a^4b^3 + 35a^3b^4 + 21a^2b^5 + 7ab^6 + b^7$.

6. $(x - y)^8$.

Ans. $x^8 - 8x^7y + 28x^6y^2 - 56x^5y^3 + 70x^4y^4 - 56x^3y^5 + 28x^2y^6 - 8xy^7 + y^8$.

7. $(a^2c^{-2} + a^{-2}c^2)^2$. *Ans.* $a^4c^{-4} + 2 + a^{-4}c^4$.

8. $(a^2 + 1 + a^{-2})^3$. *Ans.* $a^6 + 3a^4 + 6a^2 + 7 + 6a^{-2} + 3a^{-4} + a^{-6}$.

9. $(a^m + x^n)^3$. *Ans.* $a^{3m} + 3a^{2m}x^n + 3a^mx^{2n} + x^{3n}$.

POLYNOMIAL SQUARES.

217. We have seen that the square of any binomial may be written without the labor of formal multiplication (**70**). Thus, if x and y represent the terms of any binomial, then

$$(x + y)^2 = x^2 + 2xy + y^2.$$

This formula for a binomial square furnishes a simple rule for writing the square of any polynomial, in the same direct manner. To deduce the method, let it be required to square the polynomial,

$$a + b + c + d + e + \dots.$$

Put $x = a$ and $y = b + c + d + e + \dots.$ Then the square of $x + y$ will be equal to the square of the given polynomial; or

$$x^2 + 2xy + y^2 = (a + b + c + d + e + \dots)^2.$$

And the three parts of the required square will be

$$x^2 = a^2 \quad \dots \dots \quad (1),$$

$$2xy = 2ab + 2ac + 2ad + 2ae + \dots \quad (2),$$

$$y^2 = (b + c + d + e + \dots)^2.$$

Now y represents a polynomial; and to obtain its square, we must proceed as at first. Thus, put $x' = b$ and $y' = c + d + e + \dots.$ Then the square of $x' + y'$ will be equal to the square of $b + c + d + e + \dots.$ And we have

$$x'^2 = b^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3),$$
$$2x'y' = 2bc + 2bd + 2be + \ldots \quad . \quad . \quad . \quad (4),$$
$$y'^2 = (c + d + e + \ldots .)^2.$$

If we proceed with the value of y'^2 as with the value of y^2, we shall finally obtain all the parts of the required square.

By inspecting equations (1), (2), (3), and (4), we perceive that the required square will assume the following general form: $(a+b+c+d+e+\ldots .)^2 = a^2 + 2a(b+c+d+e+\ldots .) + b^2 + 2b(c+d+e+\ldots .) + c^2 + 2c(d+e+\ldots .)$, and so on. Hence, to obtain the square of any polynomial, we have the following

RULE.—*Write the square of each term, together with twice the product of each term, by the sum of all the terms which follow it, and reduce the result if necessary.*

EXAMPLES FOR PRACTICE.

1. Square $a+b+c$. *Ans.* $a^2 + 2ab + 2ac + b^2 + 2bc + c^2$.
2. Find the square of $a + b + c + d$.
   Ans. $a^2+2ab+2ac+2ad+b^2+2bc+2bd+c^2+2cd+d^2$.
3. Find the square of $a + b + c + d + e$.
   Ans. $a^2 + 2ab + 2ac + 2ad + 2ae + b^2 + 2bc + 2bd + 2be + c^2 + 2cd + 2ce + d^2 + 2de + e^2$.
4. Square $x - y + z$. *Ans.* $x^2-2xy+2xz+y^2-2yz+z^2$.
5. Find the square of $a - 2b + 3ab - c$.
   Ans. $a^2-4ab+6a^2b-2ac+4b^2-12ab^2+4bc+9a^2b^2-6abc+c^2$.
6. Find the square of $1 - a + a^2 - a^3$.
   Ans. $1 - 2a + 3a^2 - 4a^3 + 3a^4 - 2a^5 + a^6$.
7. Find the square of $3ax + 2a^2 - 4x^2 - 5$.
   Ans. $12a^3x - 24ax^3 - 30ax + 4a^4 - 7a^2x^2 - 20a^2 + 16x^4 + 40x^2 + 25$.
8. Find the square of $1 - 2x - y^2 + xy - x^2$.
   Ans. $1 - 4x - 2y^2 + 2xy + 2x^2 + 4xy^2 - 4x^2y + 4x^3 + y^4 - 2xy^3 - 2x^3y + 3x^2y^2 + x^4$.

218. In a future section we shall give a formula, called the Binomial Formula, by means of which any power of a binomial may be obtained without the labor of multiplication.

EVOLUTION.

219. A ***Root*** of any quantity is one of the equal factors which, multiplied together, will produce the given quantity.

220. The name or degree of a root corresponds to the number of equal factors into which the quantity is supposed to be divided. Thus,

The square root of a is one of the two equal factors whose product is a.

The cube root of a is one of the three equal factors whose product is a.

The fourth root of a is one of the four equal factors whose product is a; and so on.

221. ***Evolution*** is the process of extracting any root of a given quantity; it is the converse of involution.

222. There are two methods of indicating evolution:

1st. By the radical sign, $\sqrt{\ }$.

When this method is employed, the name or degree of the root is denoted by a figure or letter written above the radical, called the *index* of the root. Thus, $\sqrt[3]{a}$ denotes the *cube root* of a; and $\sqrt[4]{a}$ denotes the *fourth root* of a. When no index is written, 2 is understood. Thus, $\sqrt{x}$ denotes the square root of x, and signifies the same as $\sqrt[2]{x}$.

2d. By fractional exponents.

To explain the origin of this method of indicating roots, we observe that a quantity is raised to any power, by multiplying its exponent by the exponent of the required power. Conversely, any root of a quantity may be obtained, by dividing the exponent of the quantity by the index of the required root. Thus, the cube root of a, or a^1, is written $a^{\frac{1}{3}}$, and the cube root of a^2 will be $a^{\frac{2}{3}}$.

Hence, a fractional exponent may be analyzed as follows:

1. *The numerator denotes the power of the quantity, whose root is to be extracted.*

2. *The denominator shows what root of that power is to be extracted.*

223. The two methods of indicating roots may be illustrated by equivalent expressions, as follows:

$\sqrt{a}$, or $a^{\frac{1}{2}}$, denotes the square root of a;

$\sqrt[3]{a}$, or $a^{\frac{1}{3}}$, " " cube " " a;

$\sqrt[n]{a}$, or $a^{\frac{1}{n}}$, " " n^{th} " " a.

And if a^m represent any power of a, then

$\sqrt{a^m}$, or $a^{\frac{m}{2}}$, denotes the square root of a^m;

$\sqrt[3]{a^m}$, or $a^{\frac{m}{3}}$, " " cube " " a^m;

$\sqrt[n]{a^m}$, or $a^{\frac{m}{n}}$, " " n^{th} " " a^m.

224. A ***Surd*** is a root which cannot be exactly obtained; as $\sqrt{2}$, $\sqrt[3]{a^2}$, or $\sqrt{a^2 - 2ab}$.

A surd is called an *irrational* quantity, while a root which can be exactly obtained is called a *rational* quantity. A root will be rational when the given quantity is a perfect power corresponding in degree to the required root; otherwise it will be a surd.

The root of a *number* which is an imperfect power, may always be obtained *approximately*. Thus, $\sqrt{6}$ is a surd; but we have

$$\sqrt{6} = 2.44, \text{ nearly; for } (2.44)^2 = 5.9536.$$

225. An ***Imaginary*** root is one which is known to be impossible on account of the *sign* of the given quantity. Thus, the square root of $-a^2$, or $\sqrt{-a^2}$, is impossible, since no quantity raised to the second power will produce $-a^2$. A root which is not imaginary is said to be *real*.

ROOTS OF MONOMIALS.

226. It has already been shown that the root of a simple algebraic quantity may be expressed by dividing the exponent of the quantity by the index of the required root (**222**). And it is evident that if the exponent of the quantity will not exactly contain the index of the required root, the result must be a surd.

227. We have seen that a quantity composed of several factors, may be raised to any power by involving each factor separately to the required power (**208**). Conversely, we may obtain the root of a quantity by extracting the root of each factor separately. Thus, if $abc \ldots . k$ represent the product of any number of factors, then

$$\sqrt[n]{abc \ldots . k} = \sqrt[n]{a}\,\sqrt[n]{b}\,\sqrt[n]{c} \ldots . \sqrt[n]{k};$$

or, with fractional exponents,

$$(abc \ldots . k)^{\frac{1}{n}} = a^{\frac{1}{n}} b^{\frac{1}{n}} c^{\frac{1}{n}} \ldots . k^{\frac{1}{n}}.$$

That is,

The n^{th} root of the product of two or more factors is equal to the product of the n^{th} roots of the factors.

228. There are certain properties of roots which depend upon the law of signs in involution :

1. *Every odd root of a quantity is real, and has the same sign as the quantity itself.*

For, any positive quantity raised to an odd power is positive; and any negative quantity raised to an odd power is negative (**210**).

2. *Every even root of a positive quantity is real, and may be either positive or negative.*

For, either a positive or a negative quantity raised to an even power is positive (**210**).

3. *Every even root of a negative quantity is imaginary.*

For, no quantity, whether positive or negative, raised to an even power, will give a negative result.

229. From the principles now established, we have the following rule for extracting the roots of monomials:

RULE.—I. *Extract the required root of the numeral coefficients for a new coefficient.*

II. *Divide the exponent of each literal factor by the index of the required root.*

III. *Prefix the double sign, $\pm$, to all even roots, and the minus sign to the odd roots of a negative quantity.*

NOTES.—1. When the required root of any factor is a surd, it may be indicated either by a fractional exponent, or by the radical sign.

2. The root of a fraction may be obtained by taking the root of the numerator and denominator separately.

EXAMPLES FOR PRACTICE.

1. What is the square root of $49a^2x^4$? *Ans.* $\pm 7ax^2$.
2. What is the square root of $25c^{10}b^2$? *Ans.* $\pm 5c^5b$.
3. What is the square root of $144a^2c^4x^2y^2$? *Ans.* $\pm 12ac^2xy$.
4. What is the cube root of $125a^3$? *Ans.* $5a$.
5. What is the cube root of $-64x^6$? *Ans.* $-4x^2$.
6. What is the cube root of $-216a^3y^9$? *Ans.* $-6ay^3$.
7. What is the cube root of $729a^6x^{12}$? *Ans.* $9a^2x^4$.
8. What is the 4th root of $256a^4x^8$? *Ans.* $\pm 4ax^2$.
9. Find the 4th root of $16a$. *Ans.* $\pm 2a^{\frac{1}{4}}$, or $\pm 2\sqrt[4]{a}$.
10. Find the cube root of $27a^2x$. *Ans.* $3a^{\frac{2}{3}}x^{\frac{1}{3}}$, or $3\sqrt[3]{a^2x}$.
11. Find the 5th root of $-32x^{10}y^4$. *Ans.* $-2x^2y^{\frac{4}{5}}$, or $-2x^2\sqrt[5]{y^4}$.
12. Find the n^{th} root of $a^{3n}b^m$. *Ans.* $a^3b^{\frac{m}{n}}$.
13. Find the square root of $81a^{-4}b^6$. *Ans.* $\pm 9a^{-2}b^3$.
14. Find the cube root of $-216a^{-3n}c^{-2}$. *Ans.* $-6a^{-n}c^{-\frac{2}{3}}$.
15. Find the 5th root of $243a^{-5}b^{-10}$. *Ans.* $3a^{-1}b^{-2}$.
16. Find the m^{th} root of $a^{mn}y^{m^2}$. *Ans.* a^ny^m.
17. Find the n^{th} root of $x^{n^2}y^{n^3}z^{n^4}$. *Ans.* $x^ny^{n^2}z^{n^3}$.
18. Required the square root of $\frac{4a^2x^4}{9a^2}$. *Ans.* $\pm \frac{2ax^2}{3a}$.
19. Required the cube root of $\frac{125a^3b^6}{8x^3y^{12}}$. *Ans.* $\frac{5ab^2}{2xy^4}$.
20. Required the square root of $\frac{25a^6}{16}$. *Ans.* $\pm \frac{5a^3}{4}$.
21. Required the n^{th} root of $\frac{a^nx^{3n}}{c^{2n}y^n}$. *Ans.* $\frac{ax^3}{c^2y}$.
22. Required the n^{th} root of $\frac{a^3}{bc}$. *Ans.* $a^{\frac{3}{n}}b^{-\frac{1}{n}}c^{-\frac{1}{n}}$.
23. Find the square root of $(a-x)^2y^4$. *Ans.* $\pm (a-x)y^2$.
24. Find the cube root of $(x-1)^3(x+1)^6$. *Ans.* $(x^2-1)(x+1)$.
25. Find the square root of $x^2y^4(x-y)^2$. *Ans.* $\pm (x^2y^2-xy^3)$.

SQUARE ROOT OF POLYNOMIALS.

230. To deduce a rule for the extraction of the square root of a polynomial, let us first observe how the square of any binomial, as $a + b$, is formed. We have

$$(a + b)^2 = a^2 + 2ab + b^2.$$

And the last two terms may be written as follows:

$$(2a + b)\,b.$$

Let us now consider how the process of involution may be reversed, and the root, $a + b$, derived from the square.

Extracting the square root of a^2, we obtain a, the first term of the root. Taking a^2 from the whole expression, we have $2ab + b^2$, or $(2a + b)\,b$, for a remainder. Dividing the first term of this remainder by $2a$, as a partial divisor, we obtain b, which we place in the root, and also at the right of the $2a$ to complete the divisor, $2a + b$. Multiplying the complete divisor by b, and subtracting the product from the dividend, we have no remainder, and the work is finished.

OPERATION.

$$\begin{array}{r|l} & a^2+2ab+b^2 \;\; \underline{\,a+b\,} \\ & \underline{a^2\qquad\qquad} \\ 2a+b & \quad 2ab+b^2 \\ & \quad \underline{2ab+b^2} \end{array}$$

By the same process continued, we may extract the square root of any quantity that is a perfect square. To establish the rule in a general manner, let

$$a + b + c + d \ldots.$$

represent any polynomial. By a previous article, the square of this polynomial consists of *the square of each term, together with twice the product of each term by the sum of all the terms which follow it* (217); and the square may be written as follows:

$$a^2 + 2ab + 2ac + 2ad \ldots. + b^2 + 2bc + 2bd \ldots. + c^2 + 2cd \ldots. + d^2 \ldots.$$

And it is evident that if the root, $a+b+c+d \ldots.$, is arranged according to the powers of some letter, the square will also be arranged according to powers of the same letter.

We may now derive the root from the square, in the following manner:

OPERATION.

$$\begin{array}{ll}
 & \underline{|a+b+c+d\ldots\ldots, \text{root.}}\\
 & a^2+2ab+2ac+2ad\ldots\ldots+b^2+2bc+2bd\ldots\ldots+c^2+2cd\ldots\ldots+d^2\ldots\ldots\\
 & \underline{a^2}\\
2a+b & 2ab+2ac+2ad\ldots\ldots+b^2+2bc+2bd\ldots\ldots+c^2+2cd\ldots\ldots+d^2\ldots\ldots\\
 & \underline{2ab \qquad\qquad\qquad +b^2}\\
2a+2b+c & +2ac+2ad\ldots\ldots \quad +2bc+2bd\ldots\ldots+c^2+2cd\ldots\ldots+d^2\ldots\ldots\\
 & \underline{2ac \qquad\qquad\qquad +2bc \qquad\qquad +c^2}\\
2a+2b+2c+d & 2ad\ldots\ldots \qquad +2bd\ldots\ldots \qquad +2cd\ldots\ldots+d^2\ldots\ldots\\
 & \underline{2ad \qquad\qquad\quad +2bd \qquad\qquad +2cd \qquad +d^2}\\
 & \ldots\ldots \qquad\qquad \ldots\ldots \qquad\qquad \ldots\ldots \qquad \ldots\ldots
\end{array}$$

We find a as in the former example, and take its square from the whole expression. We then divide the first term of the remainder by $2a$, and write the quotient, b, in the root, and also in the divisor. We then multiply the complete divisor by b, subtract the product from the first remainder, and thus obtain a new dividend. Then writing $2a + 2b$ for a partial divisor, we find c in the same manner as we found b; and thus we continue till the work is finished.

If we examine the several subtrahends, taking the terms *diagonally* in the operation, we shall find a^2, $2ab$, $2ac$, $2ad$, etc.; b^2, $2bc$, $2bd$, etc.; c^2, $2cd$, etc.; d^2, etc. That is, we have, in the operation, *the square of each term of the root, together with twice the product of each term by all the terms which follow it.* Thus we have exactly *reversed* the process of forming a polynomial square. Hence the following general

RULE.—I. *Arrange the terms according to the powers of some letter, and write the square root of the first term for the first term of the root.*

II. *Subtract the square of the root thus found from the given quantity, and bring down two or more terms for a dividend.*

III. *Divide the first term of the dividend by twice the root already found, and write the result both in the root and in the divisor.*

IV. *Multiply the divisor, thus completed, by the term of the root last found, subtract the product from the dividend, and proceed with the remainder, if any, as before.*

NOTE.—According to the law of signs in evolution, every square root obtained will still be a root, if the signs of all its terms be changed.

EXAMPLES FOR PRACTICE.

1. What is the square root of $a^2+2ab+2ac+b^2+2bc+c^2$? *Ans.* $a+b+c$.

2. What is the square root of $a^4-6a^2b+4a^2+9b^2-12b+4$? *Ans.* a^2-3b+2.

3. What is the square root of $x^6+4x^5+2x^4-2x^3+5x^2-2x+1$? *Ans.* x^3+2x^2-x+1.

4. What is the square root of $1-2a+3a^2-4a^3+3a^4-2a^5+a^6$? *Ans.* $1-a+a^2-a^3$.

5. What is the square root of $4a^4b^2-12a^3b^2+8a^3b^3+9a^2b^2-12a^2b^3+4a^2b^4$? *Ans.* $2a^2b-3ab+2ab^2$.

6. What is the square root of $9x^6-30x^5y+x^4y^2+76x^3y^3-44x^2y^4-48xy^5+36y^6$? *Ans.* $3x^3-5x^2y-4xy^2+6y^3$.

7. What is the square root of $a^4-6a^2bc+4a^2cd-2a^2d^2+9b^2c^2-12bc^2d+6bcd^2+4c^2d^2-4cd^3+d^4$. *Ans.* $a^2-3bc+2cd-d^2$.

8. What is the square root of $a^4-a^3b+\frac{3a^2b^2}{4}-\frac{ab^3}{4}+\frac{b^4}{16}$?

Ans. $a^2-\frac{ab}{2}+\frac{b^2}{4}$.

9. What is the square root of $x^8-6x^5+11x^2-6x^{-1}+x^{-4}$? *Ans.* x^4-3x+x^{-2}.

10. What is the square root of $a^2b^{-2}-10ab^{-1}+27-10a^{-1}b+a^{-2}b^2$? *Ans.* $ab^{-1}-5+a^{-1}b$.

11. What is the square root of $a^{4m}+6a^{3m}c^n+11a^{2m}c^{2n}+6a^mc^{3n}+c^{4n}$? *Ans.* $a^{2m}+3a^mc^n+c^{2n}$.

SQUARE ROOT OF NUMBERS.

231. In order to discover the process of extracting the square root of a number, it is necessary to determine

1st. The relative number of places in a number and its square root.

2d. The local relations of the several figures of the root to the periods of the number.

3d. The law by which the parts of a number are combined in the formation of its square.

232. The relative number of places in a given number and its square root may be shown by illustrations, as follows:

$$\begin{array}{rr@{\qquad\qquad}rr}
1^2 = & 1 & 1^2 = & 1 \\
9^2 = & 81 & 10^2 = & 1,00 \\
99^2 = & 98,01 & 100^2 = & 1,00,00 \\
999^2 = & 99,80,01 & 1000^2 = & 1,00,00,00.
\end{array}$$

From these examples we perceive that a root consisting of 1 place may have 1 or 2 places in the square; and that in all cases the addition of 1 place to the root adds 2 places to the square. Hence,

If we point off a number into two-figure periods, commencing at the right hand, the number of periods will indicate the number of places in the square root.

233. If any number, as 2345, be decomposed at pleasure, the squares of the parts, beginning with the highest order, will be related in local value as follows:

$$\begin{array}{rl}
2000^2 = & 4\ 00\ 00\ 00 \\
2300^2 = & 5\ 29\ 00\ 00 \\
2340^2 = & 5\ 47\ 56\ 00 \\
2345^2 = & 5\ 49\ 90\ 25.
\end{array}$$

Hence,

The square of the first figure of the root is contained wholly in the first period of the power; the square of the first two figures of the root is contained wholly in the first two periods of the power; and so on.

234. If the figures of a number be separated into two parts, and written with their local value, we may then form the square of the number by the formula for a binomial square. Thus,

$76 = 70 + 6$. And if we put $a = 70$ and $b = 6$, then $a + b = 76$; and we shall have

$$\begin{array}{rr}
a^2 = & 4900 \\
2ab = & 840 \\
b^2 = & 36 \\
\hline
a^2 + 2ab + b^2 = & 5776 = 76^2.
\end{array}$$

Hence, the binomial square may be used as a formula for extracting the square root of a number.

1. Let it be required to extract the square root of 5776.

There are two periods in the number, indicating that there will be two places in the root. As the square of the tens is contained wholly in the first period (**233**), we first seek the greatest perfect square in 57. This we find to be 49, the root of which is 7, the first figure of the root sought. Hence we have $a = 70$, and subtracting a^2, or 4900, from the entire number, we have 876 for a remainder, which must be equal to $(2a + b)\,b$ (**230**). Dividing the remainder by the partial divisor, $2a$, or 140, we have $b = 6$, the second figure of the root. Completing the divisor, we have $2a + b = 146$; whence $(2a + b) \times b = 876$, and the work is complete.

OPERATION.

		57 76	76
a^2,		49 00	
$2a$,	140	8 76	
$2a + b$,	146	8 76	

It is obvious that we may omit ciphers, and still employ the figures with their proper local values, in the operation. It will not then be necessary to form the partial divisor separate from the complete divisor.

If the given number consists of more than two periods, we may find the two superior figures of the root from the first two periods (**233**), bringing down another period to the remainder. Then a in the binomial formula will represent the part of the root already found, *considered as tens of the next inferior order;* and so on.

2. Required the square root of 226576.

OPERATION.

	22 65 76	476, *Ans.*
	16	
87	665	
	609	
946	5676	
	5676	

Having found 47, the square root of the first two periods, we bring down the last period, and have 5676 for a new dividend.

We then take $2a = 47 \times 2 = 94$ for a partial divisor, whence we obtain $b = 6$, the last figure of the root. We should observe that by simply doubling the 7 in the 87, we may obtain 94, the new trial-divisor.

From these principles and illustrations, we have the following

RULE.—I. *Point off the given number into periods of two figures each, counting from unit's place toward the left and right.*

II. *Find the greatest square number in the left-hand period, and write its root for the first figure in the root sought; subtract the square number from the left-hand period, and to the remainder bring down the next period for a dividend.*

III. *At the left of the dividend write twice the first figure of the root, for a trial divisor; divide the dividend, exclusive of its right-hand figure, by the trial divisor, and write the quotient for a trial figure in the root.*

IV. *Annex the trial figure of the root to the trial divisor for a complete divisor; multiply the complete divisor by the trial figure in the root, subtract the product from the dividend, and to the remainder bring down the next period for a new dividend.*

V. *Take the last complete divisor, doubling its right-hand figure, for a new trial divisor, with which proceed as before, till the work is finished.*

NOTES.—1. If there is a remainder after all the periods have been brought down, annex periods of ciphers, and continue the operation to as many decimal places as are required.

2. If the denominator of a fraction is not a perfect square, the fraction may be first reduced to a decimal, and its root then taken.

EXAMPLES FOR PRACTICE.

1. What is the square root of 7225? *Ans.* 85.
2. What is the square root of 108241? *Ans.* 329.
3. What is the square root of 651249? *Ans.* 807.
4. What is the square root of 974169? *Ans.* 987.
5. What is the square root of 5098564? *Ans.* 2258.

6. What is the square root of 6634.1025? *Ans.* 81.45.
7. What is the square root of 1812886084? *Ans.* 42578.
8. What is the square root of .339889? *Ans.* .583.
9. What is the square root of .00524176? *Ans.* .0724.
10. What is the square root of 477? *Ans.* 21.8403+.
11. What is the square root of 11.09? *Ans.* 3.33016+.
12. Required the square root of $\frac{1369}{11881}$. *Ans.* $\frac{37}{109}$.
13. Required the square root of $\frac{1}{102030201}$. *Ans.* $\frac{1}{10101}$.
14. Required the square root of $\frac{49}{144}$. *Ans.* $\frac{7}{12}$.
15. Required the square root of $5\frac{4}{7}$. *Ans.* 2.3604—.

CONTRACTED METHOD.

235. When the required root is a surd, the work may be abridged by the method of contracted decimal multiplication. To insure a correct result, each contracted divisor should contain at least one *redundant place*—that is, one place more than is necessary to produce the required order of units in the product. This figure should be multiplied mentally, and the tens (increased by 1 when the units are 5 or more) carried to the product of the next figure.

To illustrate this principle, let it be required to divide 28337 by 53194, correct to 3 decimal places.

53194)	28337 (.5327
	26597
5319	1740
	1595
532	145
	106
53	39
	37

In multiplying the first divisor, of which the last figure, 4, is treated as redundant, we say 5 times 4 are 20, and reserve the two tens for the next partial product; then, 5 times 9 are 45, and 2 tens added make 47, and we write the unit figure of this result for the first in the contracted product. In multiplying the second divisor, 5319, we have $9 \times 3 = 27$; hence there will be three tens to carry, because 27 is nearer 30 than 20. The third divisor is 532, one unit being carried to 531 of the preceding divisor, because the rejected figure, 9, is greater than 5.

1. Required the square root of 7.12 correct to six decimal places.

We continue the operation as usual until we have obtained the dividend, 1776. At this point we omit the period of ciphers, and consider 533 as the divisor; and in multiplying by 3, the new root figure, we carry the 1 ten from the product of the redundant figure 6, and 1 also from the 8 units in this product, making 1601 for the first contracted product. After this we drop one figure from the right, to form each successive divisor, and thus continue till the work is finished.

OPERATION.

	2.668333 ± *Ans.*
	7.120000
	4
46	312
	276
526	3600
	3156
5328	44400
	42624
5336	1776*
	1601
534	175
	160
53	15
	16

It will be observed that the number of places in the root is equal to the number of places assumed in the power.

From this illustration, we have the following

RULE.—I. *If necessary, annex periods of ciphers to the given number, and assume as many figures as there are places required in the root; then proceed in the usual manner until all the assumed figures have been brought down.*

II. *Form the next trial divisor as usual, but omit to annex to it the trial figure of the root, reject one figure from the right to form each subsequent divisor, and in multiplying regard the right-hand figure of each contracted divisor as redundant.*

NOTE.—If a rejected figure is 5 or more, increase the next figure at the left by 1.

EXAMPLES.

1. Find the square root of 56 correct to 7 decimal places.
Ans. 7.4833147+.

2. Find the square root of 14 correct to 7 decimal places.
Ans. 3.7416573+.

3. Find the square root of 18 correct to 4 decimal places.
Ans. 4.2426+.

4. Find the square root of 19 correct to 6 decimal places.
Ans. 4.358898+.

5. Find the square root of 52.463 correct to 7 decimal places.
Ans. 7.2431346+.

6. Find the square root of 7 correct to 8 decimal places.
Ans. 2.64575131+.

7. Find the value of $5^{\frac{3}{2}}$ correct to 5 decimal places.
Ans. 11.18034−.

CUBE ROOT OF POLYNOMIALS.

236. We may deduce a rule for extracting the cube root of a polynomial in a manner similar to that pursued in square root, by analyzing the combination of terms in the binomial cube.

If the binomial, $a + b$, be cubed, we have

$$a^3 + 3a^2b + 3ab^2 + b^3.$$

We will now consider how the process may be reversed, and the root extracted from the power. We observe

1st. That the first term of the root may be obtained by taking the cube root of the first term of the power. Thus,

$$\sqrt[3]{a^3} = a.$$

2d. The second term of the root may be found by dividing the second term of the power by three times the square of the first term of the root. Thus,

$$3a^2b \div 3a^2 = b.$$

3d. The last three terms of the power may be factored, and written as follows:

$$(3a^2 + 3ab + b^2)\,b \quad \text{or} \quad \left\{3a^2 + (3a + b)\,b\right\} b.$$

Thus we see that if to the trial divisor, $3a^2$, we add a correction, $3ab + b^2$, or $(3a + b)\,b$, the result will be a complete divisor, which multiplied by b, will give the last three terms of the power.

Hence, the whole operation of extracting the root, $a + b$, from the cube, $a^3 + 3a^2b + 3ab^2 + b^3$, may be written as follows:

OPERATION.

$$\begin{array}{ll|lr|l}
 & & & a^3+3a^2b+3ab^2+b^3 & \underline{a+b} \\
 & & & a^3 & \\
\hline
 & & 3a^2 & 3a^2b+3ab^2+b^3 & \\
3a+b & 3ab+b^2 & 3a^2+3ab+b^2 & 3a^2b+3ab^2+b^3 & \\
\hline
\end{array}$$

Having found a, the first term of the root, we take its cube from the whole expression, and obtain $3a^2b+3ab^2+b^3$. Dividing the first term of this remainder by $3a^2$, we obtain b, the second term of the root. To complete the divisor, we first write the quantity $3a + b$; and multiplying this by b, we have $3ab + b^2$, which added to the trial divisor, gives $3a^2 + 3ab + b^2$, the complete divisor. Multiplying this by b, and subtracting the product from the dividend, there is no remainder, and the work is complete.

237. To recapitulate, we may designate the quantities employed in the foregoing operation, as follows:

$$\left.\begin{array}{lr}
\text{Trial divisor,} & 3a^2 \\
\text{First factor of correction,} & 3a + b \\
\text{Correction of trial divisor,} & 3ab + b^2 \\
\text{Complete divisor,} & 3a^2 + 3ab + b^2
\end{array}\right\} \ldots (a).$$

238. Next, suppose there are three terms in the root, as $a + b + c$.

Assume $s = a + b$; then $s + c = a + b + c$; and we have

$$(s + c)^3 = s^3 + 3s^2c + 3sc^2 + c^3.$$

If we proceed as in the last example, we shall obtain $a + b$, or that part of the root represented by s, and subtract its cube from the whole expression. There will then be left $3s^2c + 3sc^2 + c^3$, which may be factored and written

$$(3s^2 + 3sc + c^2)c \quad \text{or} \quad \left\{3s^2 + (3s + c)\,c\right\} c.$$

And we perceive that $3s^2$ will be the new trial divisor to obtain c, and that $(3s + c)\,c$ will be the new correction.

The value of $3s^2$, or $3\,(a + b)^2$, may be obtained by multiplication. It will be more convenient, however, to derive it by the addition of three quantities already used in the operation. Thus,

$$\left.\begin{array}{lr}\text{Last complete divisor,} & 3a^2 + 3ab + b^2 \\ \text{Last correction,} & 3ab + b^2 \\ \text{Square of last term of the root,} & b^2\end{array}\right\} \ldots (b).$$

$$3s^2 = 3\,(a + b)^2 = 3a^2 + 6ab + 3b^2.$$

Let it now be required to find the cube root of the polynomial

$$x^6 + 3x^5 - 3x^4 - 11x^3 + 6x^2 + 12x - 8.$$

OPERATION.

			x^2+x-2, root.
			$x^6+3x^5-3x^4-11x^3+6x^2+12x-8$
			x^6
		$3x^4$	$3x^5-3x^4-11x^3+6x^2+12x-8$
$3x^2+x$	$3x^3+x^2$	$3x^4+3x^3+x^2$	$3x^5+3x^4+x^3$
		$3x^4+6x^3+3x^2$	$-6x^4-12x^3+6x^2+12x-8$
$3x^2+3x-2$,	$-6x^2-6x+4$	$3x^4+6x^3-3x^2-6x+4$,	$-6x^4-12x^3+6x^2+12x-8$

Having arranged the polynomial according to the exponents of x, we proceed as in the former example, and obtain x^2, the first term of the root, $3x^5-3x^4-11x^3+6x^2+12x-8$ the first remainder, $3x^4$ the trial divisor, and x the second term of the root. To complete the trial divisor according to formula (a), we write three times the first term of the root plus the second, or $3x^2+x$, for the first factor of the correction. Whence we have $(3x^2+x)x$, or $3x^3+x^2$, for the correction; $3x^4+3x^3+x^2$ for the complete divisor; $(3x^4 + 3x^3 + x^2)x$, or $3x^5 + 3x^4 + x^3$, for the product; and $-6x^4 - 12x^3 + 6x^2 + 12x - 8$ for the new dividend.

To form the new trial divisor according to formula (b), we have $(3x^4+3x^3+x^2) + (3x^3+x^2) + x^2 = 3x^4+6x^3+3x^2$; whence, by division, we obtain -2 for the third term of the root. To complete the new trial divisor, we have for the first factor of the correction, $3\,(x^2+x) - 2 = 3x^2 + 3x - 2$. This may be obtained in the operation from the former factor $3x^2 + x$, by simply multiplying its second term by 3, and annexing the -2. We now find the correction, complete divisor, and product as before, and the work is finished. It is evident that three or more terms of the root will sustain the same relation to the next succeeding term, that the first sustains to the second, or the first and second to the third.

239. From the foregoing analysis we derive the following

RULE.—I. *Arrange the polynomial according to the powers of some letter, and write the cube root of the first term for the first term of the root; subtract the cube of the root thus found from the polynomial, and arrange the remainder for a dividend.*

II. *At the left of the dividend write three times the square of the root already found, for a trial divisor; divide the first term of the dividend by this divisor, and write the quotient for the next term of the root.*

III. *To three times the first term of the root annex the last term, and write the result at the left, and one line below, the trial divisor; multiply this result by the last term of the root, for a correction of the trial divisor; add the correction, and the result will be the complete divisor.*

IV. *Multiply the complete divisor by the last term of the root, subtract the product from the dividend, and arrange the remainder for a new dividend.*

V. *Add together the last complete divisor, the last correction, and the square of the last term of the root, for a new trial divisor; and by division obtain another term of the root.*

VI. *Take the first factor of the last correction with its last term multiplied by 3, and annex to it the last term of the root, for the first factor of the new correction; with which proceed as before, till the work is finished.*

EXAMPLES FOR PRACTICE.

1. What is the cube root of $27a^3 + 108a^2 + 144a + 64$?
Ans. $3a + 4$.

2. What is the cube root of $x^6 + 6x^5 - 40x^3 + 96x - 64$?
Ans. $x^2 + 2x - 4$.

3. What is the cube root of $8x^6 - 36x^5 + 66x^4 - 63x^3 + 33x^2 - 9x + 1$?
Ans. $2x^2 - 3x + 1$.

4. What is the cube root of $a^6 + 9a^5b + 24a^4b^2 + 9a^3b^3 - 24a^2b^4 + 9ab^5 - b^6$?
Ans. $a^2 + 3ab - b^2$.

5. What is the cube root of $a^9 - 6a^8 + 27a^7 - 74a^6 + 159a^5 - 234a^4 + 257a^3 - 174a^2 + 60a - 8$?
Ans. $a^3 - 2a^2 + 5a - 2$.

6. What is the cube root of $x^9 - 3x^8 + 6x^7 - 10x^6 + 12x^5 - 12x^4 + 10x^3 - 6x^2 + 3x - 1$? *Ans.* $x^3 - x^2 + x - 1$.

7. What is the cube root of $8a^3 - 12a^3b + 36a^2bc + 6a^3b^2 - 36a^2b^2c - a^3b^3 + 54ab^2c^2 + 9a^2b^3c - 27ab^3c^2 + 27b^3c^3$?

Ans. $2a - ab + 3bc$.

8. What is the cube root of $x^6 - 12x^5 + \frac{195x^4}{4} - 70x^3 + \frac{195x^2}{16} - \frac{3x}{4} + \frac{1}{64}$? *Ans.* $x^2 - 4x + \frac{1}{4}$.

9. What is the cube root of $x^9 + 6x^8 - 64x^6 - 96x^5 + 192x^4 + 512x^3 - 768x - 512$? *Ans.* $x^3 + 2x^2 - 4x - 8$.

CUBE ROOT OF NUMBERS.

240. To establish a rule for extracting the cube root of a number, we must first ascertain the relative number of places in a cube and its root. This relation is exhibited in the following examples:

Roots.	Cubes.	Roots.	Cubes.
1	1	1	1
9	729	10	1,000
99	970,299	100	1,000,000
999	997,002,999	1000	1,000,000,000.

Thus we perceive that a number consisting of *one* place, may have from *one to three* places in its cube; and that in all cases the addition of *one* place to the root adds *three* places to the cube. Hence,

If a number be pointed off into three-figure periods, commencing at units' place, the number of periods will indicate the number of places in the root.

241. To ascertain how the several figures of the root are related in local value to the periods of the power, we may decompose any number, as 5423, and form the cubes of its several parts, as follows:

$$5000^3 = 125\ 000\ 000\ 000$$
$$5400^3 = 157\ 464\ 000\ 000$$
$$5420^3 = 159\ 220\ 088\ 000$$
$$5423^3 = 159\ 484\ 621\ 967.$$

Hence,

The cube of the first figure of the root is contained wholly in the first period of the power; the cube of the first two figures of the root is contained wholly in the first two periods of the power; and so on.

242. To employ the binomial cube as a formula for extracting the cube root of a number, we must represent the first figure or figures of the root, taken with their local value, by a, and the remaining figures by b. The operation will then be the same, in form and principle, as that employed in extracting the cube root of algebraic quantities.

1. Let it be required to find the cube root of 164,206,490,176.

OPERATION.

			164206490176 | 5476
			125
		7500	39206
154	616	8116	32464
		874800	6742490
1627	11389	886189	6203323
		89762700	539167176
16416	98496	89861196	539167176

There are four periods in the given number, indicating that there will be four figures in the root. As the cube of the first figure will be contained wholly in the first period (**241**), we seek the greatest perfect cube in 164. This we find to be 125; its root is 5, which we write as the first figure of the root sought.

We may now consider the 5 as tens of the next inferior order in the root, and let $a=50$, and b represent the next figure. And since the cube of $a+b$ will be contained wholly in the first two periods of the number (**241**), we subtract a^3, or 125, from 164, and to the remainder bring down the next period, making 39206. Then this result must contain at least $3a^2b+3ab^2+b^3$ (**236**), and we therefore divide it by $3a^2$, or 7500, as a trial divisor, and obtain 4 for the value of b, or the second figure of the root.

To complete the divisor, we have $3a+b=154$ for the first factor of the correction, and $(3a+b)\,b=616$ for the first correction; whence by addition we obtain 8116, the complete divi-

sor. Multiplying this by 4, and subtracting the product from the dividend, we have, after bringing down the next period, 6742490 for a new dividend.

We may now form a new trial divisor (238, *b*). We shall have $8116+616+16=8748$; or 874800, if we give to the figures their local value with respect to the lowest order in the dividend. By division, we have 7 for the next figure of the root. To find a correction for the new trial divisor, we annex the last figure, 7, to 3 times the former figures of the root, and obtain 1627 for the first factor; and we then continue the operation, repeating the former steps, till the work is finished.

Hence we have the following

RULE.—I. *Point off the given number into periods of three figures each, counting from units' place toward the left and right.*

II. *Find the greatest cube in the left-hand period, and place its root for the first figure of the required root; subtract this cube from the first period, and to the remainder bring down the next period for a dividend.*

III. *At the left of the dividend write three times the square of the root already found, and annex two ciphers, for a trial divisor; divide the dividend, and write the quotient for the next figure of the root.*

IV. *To three times the first figure of the root annex the last; multiply this result by the last root figure, for a correction to the trial divisor; add the correction, and the result will be the complete divisor.*

V. *Multiply the complete divisor by the last figure of the root, subtract the product from the dividend, and to the remainder bring down another period for a new dividend.*

VI. *Add together the last complete divisor, the last correction, and the square of the last figure of the root, and annex two ciphers, for a new trial divisor; then by division obtain another figure of the root.*

VII. *Take the first factor of the last correction, multiplying its right-hand figure by 3, and annex the last figure of the root, for the first factor of the new correction; with which proceed as in the former steps, till the work is finished.*

EXAMPLES FOR PRACTICE.

1. Find the cube root of 148877. *Ans.* 53.
2. Find the cube root of 571787. *Ans.* 83.
3. Find the cube root of 256047875. *Ans.* 635.
4. Find the cube root of 354894912. *Ans.* 708.
5. Find the cube root of 11852.352. *Ans.* 22.8.
6. Find the cube root of 144125083907. *Ans.* 5243.
7. Find the cube root of 128100283921. *Ans.* 5041.
8. Find the cube root of 105555569176. *Ans.* 4726.
9. Find the cube root of 731189187729. *Ans.* 9009.
10. Find the cube root of 1762.790912. *Ans.* 12.08.
11. Find the cube root of 1061520150601. *Ans.* 10201.
12. Find the cube root of 33212361.641984. *Ans.* 321.44.
13. Find the cube root of 1371737997260631. *Ans.* 111111.
14. Find the cube root of .171467. *Ans.* .55555+.
15. Find the cube root of .004235801032. *Ans.* .1618.

CONTRACTED METHOD.

243. In applying the method of contracted decimal division to the extraction of the cube root of a number, we observe,

1st. For each new figure in the root, the terms in the operation extend to the right 3 places in the column of dividends, 2 places in the column of divisors, and one place in the extreme left-hand column. Hence,

2d. If at any point in the operation we omit to bring down new periods in the dividend, we must *shorten* each succeeding divisor 1 place, and each succeeding term in the left-hand column 2 places.

3d. If, however, for the first contraction in the column of divisors, and in the left-hand column, we simply omit the *extended part,* and afterward contract according to the precept just given, each contracted multiplicand will have one *redundant figure.*

1. Find the cube root of 850 correct to 8 decimal places.

OPERATION.

			9.47268237 +, root.
			850.000000
			729
274	1096	24300	121000
		25396	101584
2827	19789	2650800	19416000
		2670589	18694123
2841	568	2690427	721877*
		2690995	538199
28	17	269156	183678
		269173	161504
		26919	22174
			21535
		2692	639
			538
		269	101
			81
		27	20
			19

We proceed in the usual manner until we reach the first contracted dividend, 721877, which is obtained in the common way, the period of ciphers being omitted. The corresponding trial divisor, with the ciphers at the right omitted, is 2690427, the right hand figure of which is redundant, being of an order lower than is required to obtain a product corresponding in local value to the contracted dividend. By division, we have 2 for a new figure in the root. To obtain a correction whose lowest figure shall be of the same order as the lowest in the trial divisor, we form the term 2841 in the common way, but omit to annex 2, the last figure in the root. Then $2841 \times 2 = 5682$, of which 568 is the part required for the correction. We then have 2690995 for a complete divisor, 538199 for a product, and 183678 for the new dividend. For the next trial divisor, we add 2690995 and 568, and reject one figure, thus obtaining 269156. The square of 2, the last root figure, is of course rejected, on account of its

inferior local value. The remaining part of the operation requires no further explanation.

It will be seen that the number of places in the root is equal to the number of places assumed in the power. Hence we have the following

RULE.—I. *Assume as many places in the power as there are places required in the root, and proceed in the usual manner till all the assumed figures have been brought down.*

II. *Form the next trial divisor as usual, omitting the ciphers at the right; and reject one place in forming each subsequent trial divisor.*

III. *In completing the first contracted divisor, omit to annex the new figure of the root to the corresponding term in the left-hand column, and reject two places in forming each succeeding term in this column.*

IV. *In multiplying, treat the right-hand figure of each contracted term as redundant, both in the column at the left, and in the column of divisors.*

NOTE.—To avoid complicating the process of contracting, it is better to use none but full periods, even if the root is thereby carried beyond the required number of places.

EXAMPLES FOR PRACTICE.

1. Find the cube root of 3 correct to 6 decimal places. *Ans.* 1.442249+.

2. Find the cube root of 7 correct to 6 decimal places. *Ans.* 1.912931+.

3. Find the cube root of 156 correct to 8 decimal places. *Ans.* 5.38321261+.

4. Find the cube root of 34786 correct to 6 decimal places. *Ans.* 32.643859+.

5. Find the cube root of 10.973937 correct to 6 decimal places. *Ans* 2.222222+.

6. Find the cube root of 1500.101520125 correct to 8 decimal places. *Ans.* 11.44740066+.

7. Find the cube root of 1.164132 correct to 6 decimal places. *Ans.* 1.051963+.

SECTION IV.

RADICAL QUANTITIES.

244. A ***Radical Quantity*** is a root merely indicated, either by the radical sign or by a fractional exponent; as $3\sqrt{a}$, $\sqrt[3]{a-b}$, $c(a+b)^{\frac{1}{3}}$, $m\sqrt[n]{x^2-y^2}$. A radical quantity may be either surd or rational.

The quantity or factor placed before a radical is its coefficient. Thus in the examples just given, 3, 1, c, and m are the coefficients of the radicals.

245. The ***Degree*** of a radical quantity is denoted by the radical index, or by the denominator of the fractional exponent. Thus,

$\sqrt{a}$, $(a-b)^{\frac{1}{2}}$ are radicals of the 2d degree;

$\sqrt[3]{x^2-y}$, $a^{\frac{1}{3}}b^{\frac{1}{3}}$ are radicals of the 3d degree;

$\sqrt[n]{ac}$, $(x+y)^{\frac{1}{n}}$ are radicals of the n^{th} degree;

246. ***Similar Radicals*** are those in which the same quantity is affected by radical signs having the same index. Thus, $4\sqrt[3]{a^2+b}$, $-\sqrt[3]{a^2+b}$, and $7(a^2+b)^{\frac{1}{3}}$ are similar radicals.

REDUCTION OF RADICALS.

Case I.

247. To reduce a radical to its simplest form.

A radical is in its *simplest form* when it contains no perfect power corresponding to the degree of the radical.

1. Reduce $\sqrt{48a^6x^3}$ to its simplest form.

We have seen that the n^{th} root of a quantity is equal to the product of the n^{th} roots of its component factors (**227**). Hence we have

$$\sqrt{48a^6x^3} = \sqrt{16a^6x^2 \times 3x} = \sqrt{16a^6x^2} \times \sqrt{3x} = 4a^3x\sqrt{3x}.$$

It will be seen that we first separate the quantity under the radical sign into two factors, one of which is a pérfect square. Then according to the principle of evolution just adduced, we have the product of two radicals, one of which, $\sqrt{16a^6x^2}$, is rational, and the other, $\sqrt{3x}$, is a surd. The expression is then reduced by extracting the root of the rational part, and multiplying it by the surd.

2. Reduce $3\sqrt[3]{8x^4y^3 - 8x^3y^4}$ to its simplest form.

Factoring as before, we have

$$\begin{aligned} 3\sqrt[3]{8x^4y^3 - 8x^3y^4} &= 3 \times \sqrt[3]{8x^3y^3} \times \sqrt[3]{x-y} \\ &= 3 \times 2xy \times \sqrt[3]{x-y} \\ &= 6xy\sqrt[3]{x-y}. \end{aligned}$$

Hence the following

RULE.—I. *Separate the factors of the quantity under the radical sign into two groups, one of which shall contain all the perfect powers corresponding in degree to the radical.*

II. *Extract the root of the rational part, multiply this root by the given coefficient, and prefix the product to the surd or irrational part.*

EXAMPLES FOR PRACTICE.

Reduce the following radicals to their simplest form:

1. $\sqrt{75}$. *Ans.* $5\sqrt{3}$.
2. $\sqrt{98a^2}$. *Ans.* $7a\sqrt{2}$.
3. $\sqrt{12x^2y}$. *Ans.* $2x\sqrt{3y}$.
4. $\sqrt[3]{54x^4}$. *Ans.* $3x\sqrt[3]{2x}$.
5. $4\sqrt[3]{108}$. *Ans.* $12\sqrt[3]{4}$.
6. $\sqrt{x^3 - a^2x^2}$. *Ans.* $x\sqrt{x - a^2}$.
7. $6\sqrt[3]{32a^3}$. *Ans.* $12a\sqrt[3]{4}$.
8. $3\sqrt{28a^3x^2}$. *Ans.* $6ax\sqrt{7a}$.
9. $\sqrt[3]{a^3 + a^3b^2}$. *Ans.* $a\sqrt[3]{1 + b^2}$.
10. $(x - y)\sqrt{2x^3 - 4x^2y + 2xy^2}$. *Ans.* $(x - y)^2\sqrt{2x}$.
11. $(a - b)\sqrt{2a^2b + 4ab^2 + 2b^3}$. *Ans.* $(a^2 - b^2)\sqrt{2b}$.
12. $5b(b^3 - b^2)^{\frac{1}{2}}$. *Ans.* $5b^2(b - 1)^{\frac{1}{2}}$.

13. $(2a^7b^5 - 3a^5b^7)^{\frac{1}{5}}$. *Ans.* $ab\,(2a^2 - 3b^2)^{\frac{1}{5}}$.

14. $\dfrac{a}{b}(a^4b^5 + a^5b^4)^{\frac{1}{3}}$. *Ans.* $a^2\,(ab^2 + a^2b)^{\frac{1}{3}}$.

15. $\sqrt{8a^{2n}x^{4m}}$. *Ans.* $2a^nx^{2m}\sqrt{2}$.

16. $\sqrt[3]{a^{4m}c^{5m}}$ *Ans.* $a^mc^m\sqrt[3]{a^mc^{2m}}$.

17. $(2x^{2m}y^m - 3x^{3m}y^{3m})^{\frac{1}{m}}$. *Ans.* $x^2y\,(2 - 3x^my^{2m})^{\frac{1}{m}}$

18. $a^{-m}c\,(a^{mn}c^{2n} - a^{2mn}c^n)^{\frac{1}{n}}$. *Ans.* $c^2\,(c^n - a^{mn})^{\frac{1}{n}}$.

248. When the quantity under the radical sign is a fraction, we may transform it in such a manner that the denominator shall be a perfect power corresponding in degree to the indicated root. Then after simplifying, the quantity remaining under the radical sign will be entire. It will generally be expedient to separate the given fraction into two factors, one of which shall be a perfect power; we may then operate upon the surd part separately.

1. Reduce $\sqrt{\frac{44}{75}}$ to its simplest form.

OPERATION.

$$\sqrt{\tfrac{44}{75}} = \sqrt{\tfrac{4}{25} \times \tfrac{11}{3}} = \sqrt{\tfrac{4}{25} \times \tfrac{33}{9}} = \sqrt{\tfrac{4}{25} \times \tfrac{1}{9} \times 33} = \tfrac{2}{15}\sqrt{33},\ \textit{Ans.}$$

2. Reduce $\sqrt[3]{\frac{5}{72}}$ to its simplest form.

OPERATION.

$$\sqrt[3]{\tfrac{5}{72}} = \sqrt[3]{\tfrac{15}{216}} = \sqrt[3]{\tfrac{1}{216} \times 15} = \tfrac{1}{6}\sqrt[3]{15},\ \textit{Ans.}$$

In like manner reduce the following:

3. $\sqrt[3]{\frac{135}{32}}$. *Ans.* $\frac{3}{4}\sqrt[3]{10}$.

4. $\sqrt[3]{\frac{25}{9}}$. *Ans.* $\frac{1}{3}\sqrt[3]{75}$.

5. $\sqrt{\frac{50}{147}}$. *Ans.* $\frac{5}{21}\sqrt{6}$.

6. $2\sqrt{\dfrac{2a}{3}}$. *Ans.* $\frac{2}{3}\sqrt{6a}$.

7. $\frac{5}{6}\sqrt{\frac{72}{245}}$. *Ans.* $\frac{1}{7}\sqrt{10}$.

8. $\dfrac{x}{a}\sqrt{\dfrac{a^3b}{x^2y^2}}$. *Ans.* $\dfrac{1}{y}\sqrt{ab}$.

CASE II.

249. To reduce a rational quantity to a radical, or to introduce a coefficient of a radical under the radical sign.

Since involution and evolution are the converse of each other, we have

$$a = \sqrt{a^2} = \sqrt[3]{a^3} = \sqrt[4]{a^4}, \text{ etc.}$$

Whence, we have also,

$$a\sqrt{b} = \sqrt{a^2} \times \sqrt{b} = \sqrt{a^2b}.$$

We have, therefore, the following

RULE.—I. To reduce a rational quantity to a radical:—*Involve it to a power denoted by the degree of the required radical, and write the result under the radical sign.*

II. To introduce the coefficient of a radical quantity under the radical sign:—*Involve it to a power denoted by the degree of the radical, and multiply the quantity under the radical by the power thus obtained.*

EXAMPLES FOR PRACTICE.

1. Reduce ab^2 to a radical of the second degree. *Ans.* $\sqrt{a^2b^4}$.
2. Reduce $5a^2xy^3$ to a radical of the 3d degree.
   Ans. $\sqrt[3]{125a^6x^3y^9}$.
3. Reduce $a - cz$ to a radical of the 4th degree.
   Ans. $(a^4 - 4a^3cz + 6a^2c^2z^2 - 4ac^3z^3 + c^4z^4)^{\frac{1}{4}}$.

Introduce the coefficients of the following radicals under the radical sign:

4. $4a\sqrt{2xy}$. *Ans.* $\sqrt{32a^2xy}$.
5. $3x^2\sqrt[3]{x-y}$. *Ans.* $\sqrt[3]{27x^7 - 27x^6y}$.
6. $(a-2b)\sqrt{2a}$. *Ans.* $\sqrt{2a^3 - 8a^2b + 8ab^2}$.
7. $\dfrac{a}{c}\left(\dfrac{c}{a} - \dfrac{a}{c}\right)^{\frac{1}{3}}$. *Ans.* $\left(\dfrac{a}{c} - \dfrac{a^3}{c^3}\right)^{\frac{1}{3}}$.
8. $x\left(\dfrac{1}{x} - \dfrac{a}{x^3} + \dfrac{a^2}{x^5} - \dfrac{a^3}{x^7}\right)^{\frac{1}{2}}$. *Ans.* $\left(x - \dfrac{a}{x} + \dfrac{a^2}{x^3} - \dfrac{a^3}{x^5}\right)^{\frac{1}{2}}$.

CASE III.

250. To reduce radicals to a common index.

It may be shown that $a^{\frac{m}{n}} = a^{\frac{mr}{nr}}$, r being any integer whatever.

Let	$x = a^{\frac{m}{n}} \ldots\ldots (1);$
involving (1) to the n^{th} power,	$x^n = a^m \ldots\ldots (2);$
" (2) " r^{th} "	$x^{nr} = a^{mr} \ldots\ldots (3);$
taking the nr^{th} root of (3),	$x = a^{\frac{mr}{nr}} \ldots\ldots (4);$
equating values of x in (1) and (4),	$a^{\frac{m}{n}} = a^{\frac{mr}{nr}}$. Hence,

1. *If both terms of a fractional exponent be multiplied or divided by the same number, the value of the expression will not be changed.*

From **223** we have	$a^{\frac{m}{n}} = \sqrt[n]{a^m},$	
and	$a^{\frac{mr}{nr}} = \sqrt[nr]{a^{mr}}.$	Hence,

2. *If the index of a radical and the exponent of the quantity under the radical sign be multiplied or divided by the same number, the value of the expression will not be changed.*

1. Reduce $(ab)^{\frac{1}{2}}$ and $(a^2x)^{\frac{1}{3}}$ to a common index.

$$\left.\begin{aligned} (ab)^{\frac{1}{2}} &= (ab)^{\frac{3}{6}} = (a^3b^3)^{\frac{1}{6}} \\ (a^2x)^{\frac{1}{3}} &= (a^2x)^{\frac{2}{6}} = (a^4x^2)^{\frac{1}{6}} \end{aligned}\right\}, \textit{ Ans.}$$

2. Reduce $\sqrt[3]{a^2c}$ and $\sqrt[4]{x^3z^2}$ to a common index.

$$\left.\begin{aligned} \sqrt[3]{a^2c} &= \sqrt[12]{a^8c^4} \\ \sqrt[4]{x^3z^2} &= \sqrt[12]{x^9z^6} \end{aligned}\right\}, \textit{ Ans.}$$

Hence, we have the following

RULE.—I. When the quantities are affected by fractional exponents :—*Reduce the given exponents to their least common denominator ; then raise each quantity to a power denoted by the numerator of its new exponent, and affect each result with a fractional exponent equal to the reciprocal of the common denominator.*

II. When the quantities are affected by radical signs:—*Find the least common multiple of the given indices for the common index required; and raise the quantity under each radical sign to a power indicated by the quotient of the new index divided by the given index.*

EXAMPLES FOR PRACTICE

1. Reduce $a^{\frac{1}{2}}$, $(cd)^{\frac{1}{3}}$, and $(a^2c)^{\frac{1}{4}}$ to a common index.

   Ans. $a^{\frac{6}{12}}$, $(c^4d^4)^{\frac{1}{12}}$, $(a^6c^3)^{\frac{1}{12}}$.

2. Reduce $(3a^2x)^{\frac{1}{3}}$, $(2ax^2)^{\frac{1}{4}}$, and $(5a^3x^5)^{\frac{1}{6}}$ to a common index.

   Ans. $(81a^8x^4)^{\frac{1}{12}}$, $(8a^3x^6)^{\frac{1}{12}}$, $(25a^6x^{10})^{\frac{1}{12}}$.

3. Reduce $(a-b)^{\frac{1}{2}}$ and $(a+b)^{\frac{2}{3}}$ to a common index.

   $$\textit{Ans.}\begin{cases}(a-b)^{\frac{3}{6}}, \text{ or } (a^3-3a^2b+3ab^2-b^3)^{\frac{1}{6}}; \\ (a+b)^{\frac{4}{6}}, \text{ or } (a^4+4a^3b+6a^2b^2+4ab^3+b^4)^{\frac{1}{6}}.\end{cases}$$

4. Reduce a, $\sqrt{ac}$, $\sqrt[3]{a^2x}$, and $\sqrt[4]{2ac^2}$ to a common index.

   Ans. $\sqrt[12]{a^{12}}$, $\sqrt[12]{a^6c^6}$, $\sqrt[12]{a^8x^4}$, $\sqrt[12]{8a^3c^6}$.

5. Reduce $\sqrt{2}$, $\sqrt[4]{2}$, and $\sqrt[5]{2}$ to a common index.

   Ans. $\sqrt[20]{1024}$, $\sqrt[20]{32}$, $\sqrt[20]{16}$.

6. Reduce a^2, $\sqrt{5x}$, $\sqrt[3]{2ax}$, and $\sqrt[4]{4a^2x}$ to a common index.

   Ans. $\sqrt[12]{a^{24}}$, $\sqrt[12]{15625x^6}$, $\sqrt[12]{16a^4x^4}$, $\sqrt[12]{64a^6x^3}$.

7. Reduce $\sqrt{x-y}$, $\sqrt[3]{x+y}$, and $\sqrt[6]{x^2-y^2}$ to a common index.

   Ans. $\sqrt[6]{(x-y)^3}$, $\sqrt[6]{(x+y)^2}$, $\sqrt[6]{x^2-y^2}$.

8. Reduce $\sqrt{ax}$, $\sqrt[m]{xy}$, and $\sqrt[n]{cx}$ to a common index.

   Ans. $\sqrt[2mn]{a^{mn}x^{mn}}$, $\sqrt[2mn]{x^{2n}y^{2n}}$, $\sqrt[2mn]{c^{2m}x^{2m}}$.

ADDITION OF RADICALS.

251. When the quantities to be added are similar radicals, it is evident that the common radical part may be made the *unit of addition;* the result will then be a single radical whose coefficient is the sum of the coefficients of the given radicals.

Radicals which do not appear to be similar, may become similar when reduced to their simplest forms.

1. What is the sum of $7\sqrt{ac}$, $3\sqrt{ac}$, and $5\sqrt{ac}$?

$$7\sqrt{ac} + 3\sqrt{ac} + 5\sqrt{ac} = (7 + 3 + 5)\sqrt{ac} = 15\sqrt{ac}, \textit{ Ans.}$$

2. What is the sum of $\sqrt[3]{8a^5c}$, $\sqrt[3]{27a^5c}$, and $\sqrt[3]{64a^2c^4}$?

OPERATION.

$$\begin{aligned} \sqrt[3]{8a^5c} &= 2a\sqrt[3]{a^2c} \\ \sqrt[3]{27a^5c} &= 3a\sqrt[3]{a^2c} \\ \sqrt[3]{64a^2c^4} &= 4c\sqrt[3]{a^2c} \\ \text{Sum} &= (5a+4c)\sqrt[3]{a^2c}, \textit{ Ans.} \end{aligned}$$

If the given radicals are dissimilar, the addition can only be indicated. Hence the following

RULE.—I. *Reduce each radical to its simplest form.*

II. *If the resulting radicals are similar, add their coefficients, and to the sum annex the common radical; if dissimilar, indicate the addition by the proper signs.*

EXAMPLES FOR PRACTICE.

1. Find the sum of $\sqrt{16a^2x}$ and $\sqrt{4a^2x}$. *Ans.* $6a\sqrt{x}$.
2. Find the sum of $\sqrt{32}$, $\sqrt{72}$, and $\sqrt{128}$. *Ans.* $18\sqrt{2}$.
3. Find the sum of $\sqrt[3]{40}$, $\sqrt[3]{135}$, and $\sqrt[3]{625}$. *Ans.* $10\sqrt[3]{5}$.
4. Find the sum of $\sqrt[3]{108}$, $9\sqrt[3]{4}$, and $\sqrt[3]{1372}$. *Ans.* $19\sqrt[3]{4}$.
5. Find the sum of $\sqrt{\frac{1}{2}}$, $\sqrt{\frac{2}{9}}$, and $\sqrt{\frac{1}{18}}$. *Ans.* $\sqrt{2}$.
6. Find the sum of $\sqrt[3]{\frac{81}{8}}$, $\sqrt[3]{\frac{81}{64}}$, and $\sqrt[3]{\frac{375}{216}}$. *Ans.* $\frac{37}{12}\sqrt[3]{3}$.
7. Find the sum of $\frac{3}{5}\sqrt{\frac{2}{3}}$, $\frac{3}{4}\sqrt{\frac{25}{6}}$, and $\frac{7}{8}\sqrt{\frac{96}{25}}$. *Ans.* $\frac{61}{40}\sqrt{6}$.
8. Find the sum of $3\sqrt{abm^2}$, $m\sqrt{4ab}$, and $\sqrt{25abm^2}$. *Ans.* $10m\sqrt{ab}$.
9. Find the sum of $2a\sqrt{c^2x - c^2y}$, $3c\sqrt{a^2x - a^2y}$, and $5\sqrt{a^2c^2x - a^2c^2y}$. *Ans.* $10ac\sqrt{x - y}$.

10. Find the sum of $\sqrt{20a^2m - 20acm + 5mc^2}$ and $\sqrt{20mc^2 - 60acm + 45a^2m}$. *Ans.* $(c - a)\sqrt{5m}$.

11. Find the sum of $3\sqrt[3]{cx^3}$, $\sqrt[3]{ax^3}$, and $2\sqrt[3]{ax^3}$.
Ans. $3x(\sqrt[3]{c} + \sqrt[3]{a})$.

12. Find the sum of $5a(cx^3 - dx^3)^{\frac{1}{3}}$ and $2x(a^3d - a^3c)^{\frac{1}{3}}$.
Ans. $3ax(c - d)^{\frac{1}{3}}$.

13. Find the sum of $\sqrt{\frac{a^2(a-b)}{a+b}}$, $\sqrt{\frac{b^2(a+b)}{a-b}}$, and $(a^2 - 3b^2)\sqrt{\frac{1}{a^2 - b^2}}$. *Ans.* $2\sqrt{a^2 - b^2}$.

14. Find the sum of $\sqrt{(1+a)^{-1}}$, $\sqrt{a^2(1+a)^{-1}}$, and $a\sqrt{(1+a)(1-a)^{-2}}$. *Ans.* $\frac{\sqrt{1+a}}{1-a}$.

SUBTRACTION OF RADICALS.

252. When the radicals are similar, it is evident that we may make the common radical the unit of subtraction. Hence the following

RULE.—I. *Reduce each radical to its simplest form.*

II. *If the resulting radicals are similar, find the difference of the coefficients, and to the result annex the common radical part; if dissimilar, indicate the subtraction by the proper sign.*

EXAMPLES FOR PRACTICE.

1. From $4\sqrt{135}$ take $2\sqrt{60}$. *Ans.* $8\sqrt{15}$.
2. From $\sqrt{75}$ take $\sqrt{50}$. *Ans.* $5(\sqrt{3} - \sqrt{2})$.
3. From $3\sqrt{16a^4b}$ take $3\sqrt{a^2b}$. *Ans.* $(12a^2 - 3a)\sqrt{b}$.
4. From $\frac{1}{2}\sqrt[3]{\frac{297}{8}}$ take $\frac{1}{4}\sqrt[3]{\frac{3773}{125}}$. *Ans.* $\frac{2}{5}\sqrt{11}$.
5. From $\frac{2}{3}\sqrt{\frac{490a^2}{338}}$ take $\frac{a}{13}\sqrt{\frac{361}{5}}$. *Ans.* $\frac{a}{15}\sqrt{5}$.

6. From $(a^2c^3 - 3c^3x)^{\frac{1}{3}}$ take $2\,(a^2d^3 - 3d^3x)^{\frac{1}{3}}$.

Ans. $(c - 2d)\,(a^2 - 3x)^{\frac{1}{3}}$.

7. From $(a^3 - ab^2 + a^2b - b^3)^{\frac{1}{2}}$ take $(a^3 - 3a^2b + 3ab^2 - b^3)^{\frac{1}{2}}$.

Ans. $2b\,(a - b)^{\frac{1}{2}}$.

8. From $a\sqrt{\dfrac{b^2x + b^2}{x - 1}}$ take $b\sqrt{\dfrac{a^2x - a^2}{x + 1}}$.

Ans. $\dfrac{2ab}{x^2 - 1}\sqrt{x^2 - 1}$.

MULTIPLICATION OF RADICALS.

253. It has already been shown (**227**) that the n^{th} root of the product of two or more factors is equal to the product of the n^{th} roots of those factors. And since the converse of this proposition is true, we shall have

$$\sqrt[n]{a} \times \sqrt[n]{b} = \sqrt[n]{ab}.$$

If the radicals have coefficients, the product of the coefficients may be taken separately. Thus,

$$c\sqrt[n]{a} \times d\sqrt[n]{b} = c \times d \times \sqrt[n]{a} \times \sqrt[n]{b} = cd\sqrt[n]{ab}.$$

If the radicals have not a common index, they must first be reduced to the same degree.

Let it be required to find the product of $a\sqrt{x}$ and $b\sqrt[3]{x^2y}$.

OPERATION.

$$a\sqrt{x} = a\sqrt[6]{x^3}$$
$$b\sqrt[3]{x^2y} = b\sqrt[6]{x^4y^2}$$

Product, $\quad ab\sqrt[6]{x^7y^2} = abx\sqrt[6]{xy^2}$, *Ans.*

Hence the following

RULE.—I. *If necessary, reduce the given radicals to a common index.*

II. *Multiply the quantities in the radical parts together, and place the product under the common radical sign; to this result prefix the product of the given coefficients, and reduce the whole to its simplest form.*

EXAMPLES FOR PRACTICE.

Find the following indicated products :

1. $5\sqrt{5} \times 3\sqrt{8}$. *Ans.* $30\sqrt{10}$.
2. $4\sqrt{12} \times 3\sqrt{2}$. *Ans.* $24\sqrt{6}$.
3. $3\sqrt{2} \times 2\sqrt{8}$. *Ans.* 24.
4. $2\sqrt{5} \times 2\sqrt{10} \times 3\sqrt{6}$. *Ans.* $120\sqrt{3}$.
5. $2\sqrt[3]{14} \times 3\sqrt[3]{4}$. *Ans.* $12\sqrt[3]{7}$.
6. $5c\sqrt{ax} \times c\sqrt[3]{a^2} \times \sqrt[3]{ax^2}$. *Ans.* $5ac^2x\sqrt[6]{a^3x}$.
7. $(xy)^{\frac{1}{2}} \times (xz)^{\frac{2}{3}} \times (yz)^{\frac{3}{4}}$. *Ans.* $xyz\,(x^2y^3z^5)^{\frac{1}{12}}$.
8. $(x-y)^{\frac{2}{3}} \times (x+y)^{\frac{3}{4}}$ *Ans.* $\sqrt[12]{(x^2-y^2)^8\,(x+y)}$.
9. $\sqrt[3]{15} \times \sqrt{10}$. *Ans.* $\sqrt[6]{225000}$.
10. $\frac{a}{b}\sqrt{\frac{x}{y}} \times \frac{y}{x}\sqrt[3]{\frac{b^2}{a^3}} \times \sqrt[3]{\frac{bx^2}{ay^2}}$. *Ans.* $\sqrt[6]{\frac{x}{y}}$.
11. $\sqrt{\frac{ax^2}{(a+x)^2}} \times \sqrt{\frac{b(a^2-x^2)^2}{x^4}} \times \sqrt[4]{\frac{a^4c}{(a-x)^4}}$. *Ans.* $\frac{a}{x}\sqrt[4]{a^2b^2c}$.

DIVISION OF RADICALS.

254. Since a fraction is raised to any power by involving its numerator and denominator separately to the required power, it is evident that any root of a fraction will be obtained by extracting the required root of each term separately. Hence we have

$$\sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}}.$$

Conversely, we shall have

$$\frac{\sqrt[n]{a}}{\sqrt[n]{b}} = \sqrt[n]{\frac{a}{b}}.$$ That is,

The quotient of the n^{th} roots of two quantities is equal to the n^{th} root of their quotient.

Upon this principle is based the rule for the division of radicals.

1. Divide $6a^2\sqrt{bc}$ by $3a\sqrt{c}$.

$$\frac{6a^2\sqrt{bc}}{3a\sqrt{c}} = \frac{6a^2}{3a}\sqrt{\frac{bc}{c}} = 2a\sqrt{b},\ \textit{Ans.}$$

2. Divide $\sqrt[3]{x^2y}$ by $\sqrt{xy}$.

$$\frac{\sqrt[3]{x^2y}}{\sqrt{xy}} = \frac{\sqrt[6]{x^4y^2}}{\sqrt[6]{x^3y^3}} = \sqrt[6]{\frac{x^4y^2}{x^3y^3}} = \sqrt[6]{\frac{x}{y}},\ \textit{Ans.}$$

Hence the following

RULE.—I. *If necessary, reduce the radicals to a common index.*

II. *Divide the coefficient of the dividend by the coefficient of the divisor; divide also the quantity in the radical part of the dividend by the quantity in the radical part of the divisor, placing the quotient under the common radical sign. Prefix the former quotient to the latter, and reduce the result to its simplest form.*

EXAMPLES FOR PRACTICE.

1. Divide $4\sqrt{50}$ by $2\sqrt{5}$. *Ans.* $2\sqrt{10}$.

2. Divide $6\sqrt[3]{100}$ by $3\sqrt[3]{5}$. *Ans.* $2\sqrt[3]{20}$.

3. Divide $\sqrt[3]{20a^2d}$ by $\sqrt{15ad}$. *Ans.* $\sqrt[6]{\dfrac{16a}{135d}}$.

4. Divide $(a^2b^2d^3)^{\frac{1}{6}}$ by $d^{\frac{1}{2}}$. *Ans.* $(ab)^{\frac{1}{3}}$.

5. Divide $(16a^3 - 12a^2x)^{\frac{1}{2}}$ by $2a$. *Ans.* $(4a - 3x)^{\frac{1}{2}}$.

6. Divide 45 by $3\sqrt{5}$. *Ans.* $3\sqrt{5}$.

7. Divide $(ab^2c^2)^{\frac{1}{3}}$ by $(a^2b^3c^4)^{\frac{1}{5}}$. *Ans.* $\sqrt[15]{\dfrac{b}{ac^2}}$.

8. Divide $12c^2(a - x)^{\frac{3}{4}}$ by $4c(a - x)^{\frac{2}{3}}$. *Ans.* $3c(a - x)^{\frac{1}{12}}$.

9. Divide $(a^3c)^{\frac{1}{m}}$ by $(ac^3)^{\frac{1}{n}}$. *Ans.* $(a^{3n-m}c^{n-3m})^{\frac{1}{mn}}$.

10. Divide $\dfrac{\sqrt{a}}{\sqrt[3]{x}}$ by $\dfrac{\sqrt[4]{a^3}}{\sqrt[3]{x^2}}$. *Ans.* $\sqrt[12]{\dfrac{x^4}{a^3}}$.

11. Divide $\sqrt[4]{a^2b - ab^2}$ by $\sqrt{ab}$. *Ans.* $\dfrac{1}{ab}\sqrt[4]{a^4b^3 - a^3b^4}$.

POWERS AND ROOTS OF RADICAL QUANTITIES.

255. According to the rule for multiplication of radicals, to form the m^{th} power of $a^{\frac{1}{n}}$, or $\sqrt[n]{a}$, we must take the quantity, a, m times as a factor, and affect the result by the common radical index. Hence,

$$(a^{\frac{1}{n}})^m = (a^m)^{\frac{1}{n}} = a^{\frac{m}{n}},$$

or $$(\sqrt[n]{a})^m = \sqrt[n]{a^m}.$$ That is,

The m^{th} power of the n^{th} root of a quantity is equal to the n^{th} root of the m^{th} power of that quantity.

256. To obtain the m^{th} root of the radical $a^{\frac{1}{n}}$, or $\sqrt[n]{a}$, we may proceed as follows: Let

$$x = (a^{\frac{1}{n}})^{\frac{1}{m}} \quad \text{or} \quad \sqrt[m]{\sqrt[n]{a}} \quad . \; . \; . \quad (1).$$

Involving both members of (1) to the m^{th} power,

$$x^m = a^{\frac{1}{n}} \quad \text{or} \quad \sqrt[n]{a} \quad . \; . \; . \; . \; . \quad (2);$$

involving both members of (2) to the n^{th} power,

$$x^{mn} = a \quad \text{or} \quad a \; . \; . \; . \; . \; . \; . \; . \quad (3);$$

taking the mn^{th} root of each member of (3),

$$x = a^{\frac{1}{mn}} \quad \text{or} \quad \sqrt[mn]{a} \quad . \; . \; . \; . \; . \quad (4);$$

hence, by equating the values of x in (1) and (4),

$$(a^{\frac{1}{n}})^{\frac{1}{m}} = a^{\frac{1}{mn}},$$

or, $$\sqrt[m]{\sqrt[n]{a}} = \sqrt[mn]{a}.$$ That is,

The m^{th} root of the n^{th} root of a quantity is equal to the mn^{th} root of that quantity.

257. Since $\sqrt[m]{\sqrt[n]{a}} = \sqrt[mn]{a}$, and $\sqrt[n]{\sqrt[m]{a}} = \sqrt[mn]{a}$, we have

$$\sqrt[m]{\sqrt[n]{a}} = \sqrt[n]{\sqrt[m]{a}}.$$ That is,

The m^{th} root of the n^{th} root of a quantity is the same as the n^{th} root of the m^{th} root of that quantity.

CASE I.

258. To involve a radical quantity to any power.

1. Raise $(ab)^{\frac{1}{4}}$ to the 2d power.

From **255** we have

$$\{(ab)^{\frac{1}{4}}\}^2 = (ab)^{\frac{2}{4}} = (ab)^{\frac{1}{2}}, \textit{ Ans.}$$

2. Raise $\sqrt[6]{2ax}$ to the 4th power.

From **255** we have

$$(\sqrt[6]{2ax})^4 = \sqrt[6]{16a^4x^4}.$$

But since $6 = 2 \times 3$, we have, from **256**,

$$\sqrt[6]{16a^4x^4} = \sqrt[3]{\sqrt{16a^4x^4}} = \sqrt[3]{4a^2x^2}, \textit{ Ans.}$$

In practice, the simplification may be effected by canceling a factor from the index of the radical, and extracting the corresponding root of the quantity under the radical sign. Thus, in general terms, we have

$$\sqrt[mr]{a^r} = \sqrt[m]{a}.$$

Hence the following

RULE.—I. *If the quantity is affected by a fractional exponent, multiply this exponent by the exponent of the required power.*

II. *If the quantity is affected by the radical sign, raise the quantity under the radical sign to the required power; and if the result is a perfect power, of a degree corresponding to any factor of the radical index, cancel this factor from the index, and extract the corresponding root of the quantity under the radical sign.*

NOTE.—The coefficient may be involved separately.

EXAMPLES FOR PRACTICE.

1. Raise $\sqrt[5]{2a^2}$ to the 3d power. *Ans.* $a\sqrt[5]{8a}$.
2. Raise $\sqrt[3]{x^2y^2}$ to the 2d power, *Ans.* $xy\sqrt[3]{xy}$.
3. Raise $3\sqrt[6]{4a^5c}$ to the 4th power. *Ans.* $162a^3\sqrt[3]{2ac^2}$.
4. Raise $(a-b)^{\frac{3}{4}}$ to the 2d power. *Ans.* $(a-b)^{\frac{3}{2}}$.

5. Raise $\sqrt[10]{12ab^2}$ to the 5th power. *Ans.* $2b\sqrt{3a}$.

6. Raise $\sqrt[12]{c\,(a-x)^2}$ to the 8th power. *Ans.* $(a-x)\sqrt[3]{c^2\,(a-x)}$.

7. Raise $ax\sqrt{ax}$ to the 4th power. *Ans.* a^6x^6.

8. Raise $\sqrt[3]{x^3y^2 - x^2y^3}$ to the 2d power. *Ans.* $xy\sqrt[3]{xy\,(x-y)^2}$.

9. Raise $(a+x)^{\frac{1}{3}}$ to the 6th power. *Ans.* $a^2 + 2ax + x^2$.

10. Raise $\frac{a}{x}\sqrt[10]{96cx^6}$ to the 2d power. *Ans.* $\frac{2a^2}{x}\sqrt[5]{3cx}$.

Case II.

259. To extract any root of a radical quantity.

1. What is the square root of $4\sqrt[3]{9a^2x^4}$?

Since the coefficient is a perfect square,

$$\sqrt{4\sqrt[3]{9a^2x^4}} = 2\sqrt{\sqrt[3]{9a^2x^4}}.$$

But from **257** we have

$$2\sqrt{\sqrt[3]{9a^2x^4}} = 2\sqrt[3]{\sqrt{9a^2x^4}} = 2\sqrt[3]{3ax^2}, \ \textit{Ans.}$$

2. What is the 6th root of $5cd^3\sqrt{5c}$?

Passing the coefficient under the radical sign, we have

$$5cd^3\sqrt{5c} = \sqrt{125c^3d^6}.$$

But by **256**, $$\sqrt[6]{\sqrt{125c^3d^6}} = \sqrt[12]{125c^3d^6}.$$

Reducing this result by canceling the factor 3 from the radical index, and taking the cube root of the quantity, we have

$$\sqrt[12]{125c^3d^6} = \sqrt[4]{5cd^2}, \ \textit{Ans.}$$

3. What is the 4th root of $(ac)^{\frac{2}{3}}$?

By **256** we have

$$\left\{(ac)^{\frac{2}{3}}\right\}^{\frac{1}{4}} = (ac)^{\frac{2}{3}\times\frac{1}{4}} = (ac)^{\frac{1}{6}}, \ \textit{Ans.}$$

Hence the following

RULE.—I. *If the quantity is affected by a fractional exponent, divide this exponent by the index of the required root.*

II. *If the quantity is affected by the radical sign, extract the required root of the quantity under the radical sign, if possible; otherwise, multiply the index of the radical by the index of the required root, and simplify the result as in Case I.*

III. *If the given radical has a coefficient, extract its root separately when possible; otherwise, pass the coefficients under the radical.*

EXAMPLES FOR PRACTICE.

1. Find the cube root of $2\sqrt{ac}$. *Ans.* $\sqrt[6]{4ac}$.
2. Find the cube root of $a\sqrt[4]{a^2x^3}$. *Ans.* $\sqrt[4]{a^2x}$.
3. Find the 4th root of $2\sqrt[3]{98}$. *Ans.* $\sqrt[6]{28}$.
4. Find the square root of $\frac{2}{3}\sqrt[5]{486}$. *Ans.* $\sqrt[5]{8}$.
5. Find the square root of $49a^4\sqrt{abx}$. *Ans.* $7a^2\sqrt[4]{abx}$.
6. Find the cube root of $5\sqrt{5}$. *Ans.* $\sqrt{5}$.
7. Find the 6th root of $\left(\frac{a^3x^6}{c^6y^3}\right)^{\frac{2}{3}}$. *Ans.* $\left(\frac{ax^2}{c^2y}\right)^{\frac{1}{3}}$.
8. Find the 4th root of $\frac{4}{9}\sqrt[3]{\frac{4}{9}}$. *Ans.* $\frac{1}{3}\sqrt[3]{12}$.

260. The principle established in **256**, viz., that

$$\sqrt[mn]{a} = \sqrt[m]{\sqrt[n]{a}} = \sqrt[n]{\sqrt[m]{a}},$$

may be conveniently applied to the extraction of the higher roots of quantities, when the index of the required root is a composite number.

EXAMPLES.

1. Required the 4th root of 8603056.

Since $4 = 2 \times 2$, we take the square root of the square root of the given number. Thus,

$$\sqrt{8603056} = 2916; \quad \sqrt{2916} = 54, \ Ans.$$

2. Required the 6th root of 117649.

Since $6 = 2 \times 3$, we have

$$\sqrt{117649} = 343; \quad \sqrt[3]{343} = 7, \ Ans.$$

3. Required the 4th root of 1296. *Ans.* 6.

4. Required the 6th root of 177978515625. *Ans.* 75.

5. Required the 6th root of 191102976. *Ans.* 24.

6. Required the 8th root of 65536. *Ans.* 4.

7. Required the 4th root of $a^4 - 8a^3b + 24a^2b^2 - 32ab^3 + 16b^4$.
Ans. $a - 2b$.

8. Required the 6th root of $a^{12} + 6a^{10}b + 15a^8b^2 + 20a^6b^3 + 15a^4b^4 + 6a^2b^5 + b^6$. *Ans.* $a^2 + b$.

GENERAL THEORY OF EXPONENTS.

261. It has already been shown that

$$a^m \times a^n = a^{m+n}, \quad \frac{a^m}{a^n} = a^{m-n}, \quad \text{and} \quad (a^m)^n = a^{mn},$$

m and n being integers, and either positive or negative. To prove that the above relations are true *universally*, it remains only to show that they are true when m and n are *fractional.*

We will therefore place

$$m = \frac{p}{q} \quad \text{and} \quad n = \frac{r}{s}.$$

I. To show that $a^{\frac{p}{q}} \times a^{\frac{r}{s}} = a^{\frac{p}{q}+\frac{r}{s}}$.

Reducing the exponents to a common denominator, we have

$$a^{\frac{p}{q}} \times a^{\frac{r}{s}} = a^{\frac{ps}{qs}} \times a^{\frac{qr}{qs}}.$$

But from the nature of fractional exponents (**222**), the second member of this equation may be written

$$(a^{ps})^{\frac{1}{qs}} \times (a^{qr})^{\frac{1}{qs}};$$

and as the two factors have the same radical index (**227**), the result reduces to

$$(a^{ps} \times a^{qr})^{\frac{1}{qs}};$$

and since ps and qr are *integral*, this last result becomes

$$(a^{ps+qr})^{\frac{1}{qs}} = a^{\frac{ps+qr}{qs}} = a^{\frac{p}{q}+\frac{r}{s}}.$$

That is,

$$a^{\frac{p}{q}} \times a^{\frac{r}{s}} = a^{\frac{p}{q}+\frac{r}{s}},$$

which was to be proved.

II. To show that $a^{\frac{p}{q}} \div a^{\frac{r}{s}} = a^{\frac{p}{q}-\frac{r}{s}}$.

By transformations similar to those just employed, we have

$$\frac{a^{\frac{p}{q}}}{a^{\frac{r}{s}}} = \frac{a^{\frac{ps}{qs}}}{a^{\frac{qr}{qs}}} = \frac{(a^{ps})^{\frac{1}{qs}}}{(a^{qr})^{\frac{1}{qs}}}$$

$$= \left(\frac{a^{ps}}{a^{qr}}\right)^{\frac{1}{qs}}$$

$$= (a^{ps-qr})^{\frac{1}{qs}}$$

$$= a^{\frac{ps-qr}{qs}} = a^{\frac{p}{q}-\frac{r}{s}},$$

which was to be proved.

III. To show that $(a^{\frac{p}{q}})^{\frac{r}{s}} = a^{\frac{pr}{qs}}$.

Let us place

$$x = (a^{\frac{p}{q}})^{\frac{r}{s}} \quad . \quad . \quad . \quad . \quad (1).$$

Involving (1) to the power denoted by s,

$$x^s = (a^{\frac{p}{q}})^r \quad . \quad . \quad . \quad . \quad (2);$$

by **255** equation (2) becomes

$$x^s = a^{\frac{pr}{q}} \quad . \quad . \quad . \quad . \quad (3);$$

involving (3) to the power denoted by q,

$$x^{qs} = a^{pr} \quad . \quad . \quad . \quad . \quad (4);$$

extracting the root whose index is qs,

$$x = a^{\frac{pr}{qs}} \quad . \quad . \quad . \quad . \quad (5);$$

hence, by equating the values of x in (1) and (5),

$$(a^{\frac{p}{q}})^{\frac{r}{s}} = a^{\frac{pr}{qs}},$$

which was to be proved.

We conclude, therefore, that in multiplication, division, involution and evolution, the same rule will apply, whether the exponents are positive or negative, integral or fractional.

EXAMPLES.

1. Multiply $a^{\frac{1}{2}}b^{\frac{2}{3}}$ by $a^{\frac{2}{3}}b^{\frac{1}{3}}$, and simplify the product.

$$a^{\frac{1}{2}}b^{\frac{2}{3}} \times a^{\frac{2}{3}}b^{\frac{1}{3}} = \left(a^{\frac{1}{2}} \times a^{\frac{2}{3}}\right) \times \left(b^{\frac{2}{3}} \times b^{\frac{1}{3}}\right) = a^{\frac{7}{6}}b = ab\sqrt[6]{a}, \textit{ Ans.}$$

2. Simplify the expression, $\left(x^{\frac{2}{3}} \times x^{\frac{2}{5}}\right)^{\frac{3}{4}}$.

$$\left(x^{\frac{2}{3}} \times x^{\frac{2}{5}}\right)^{\frac{3}{4}} = \left(x^{\frac{16}{15}}\right)^{\frac{3}{4}} = x^{\frac{4}{5}}, \textit{ Ans.}$$

3. Multiply $x^{\frac{3}{4}} - 3x^{\frac{1}{2}} + x^{\frac{1}{4}}$ by $x^{\frac{1}{2}} - 2x^{\frac{1}{4}} - 3$.

OPERATION.

$$\begin{array}{llllll}
x^{\frac{3}{4}} & - 3x^{\frac{1}{2}} & + x^{\frac{1}{4}} & & & \\
x^{\frac{1}{2}} & - 2x^{\frac{1}{4}} & - 3 & & & \\
\hline
x^{\frac{5}{4}} & - 3x & + x^{\frac{3}{4}} & & & \\
 & - 2x & + 6x^{\frac{3}{4}} & - 2x^{\frac{1}{2}} & & \\
 & & - 3x^{\frac{3}{4}} & + 9x^{\frac{1}{2}} & - 3x^{\frac{1}{4}} & \\
\hline
x^{\frac{5}{4}} & - 5x & + 4x^{\frac{3}{4}} & + 7x^{\frac{1}{2}} & - 3x^{\frac{1}{4}}, & \textit{Ans.}
\end{array}$$

4. Divide $x - 5\sqrt[6]{x^5} + 7\sqrt[3]{x^2} - 5\sqrt{x} - 6\sqrt[3]{x}$ by $\sqrt[3]{x^2} - 2\sqrt{x} + 3\sqrt[3]{x}$.

OPERATION.

$$\begin{array}{l|l}
x - 5\sqrt[6]{x^5} + 7\sqrt[3]{x^2} - 5\sqrt{x} - 6\sqrt[3]{x} & \sqrt[3]{x^2} - 2\sqrt{x} + 3\sqrt[3]{x} \\
\cline{2-2}
x - 2\sqrt[6]{x^5} + 3\sqrt[3]{x^2} & \sqrt[3]{x} - 3\sqrt[6]{x} - 2, \textit{ Ans.} \\
\cline{1-1}
\quad - 3\sqrt[6]{x^5} + 4\sqrt[3]{x^2} - 5\sqrt{x} & \\
\quad - 3\sqrt[6]{x^5} + 6\sqrt[3]{x^2} - 9\sqrt{x} & \\
\cline{1-1}
\qquad\qquad - 2\sqrt[3]{x^2} + 4\sqrt{x} - 6\sqrt[3]{x} & \\
\qquad\qquad - 2\sqrt[3]{x^2} + 4\sqrt{x} - 6\sqrt[3]{x} & \\
\cline{1-1}
\end{array}$$

5. Multiply $x^{\frac{1}{2}}$ by $x^{\frac{1}{3}}$. *Ans.* $x^{\frac{5}{6}}$.

6. Multiply $a^2b^{\frac{1}{4}}$ by $a^{\frac{1}{2}}b^{\frac{1}{4}}$. *Ans.* $a^2\sqrt{ab}$.

7. Find the product of $a^{\frac{1}{4}}$, $a^{\frac{1}{2}}$, $a^{\frac{2}{3}}$, and $a^{-\frac{3}{4}}$. *Ans.* $\sqrt[3]{a^2}$.

8. Divide $a^{\frac{1}{2}}c^{\frac{2}{3}}$ by $a^{\frac{2}{3}}c^{\frac{1}{3}}$. *Ans.* $\left(\frac{c^2}{a}\right)^{\frac{1}{6}}$.

9. Divide $\left(x^{\frac{3}{4}}\right)^{\frac{2}{3}}$ by $\left(x^{\frac{1}{2}}\right)^{\frac{4}{3}}$. *Ans.* $\left(\frac{1}{x}\right)^{\frac{1}{6}}$.

10. Multiply $a^{\frac{1}{2}} - a^{\frac{1}{4}}$ by $a^{\frac{1}{4}} + 1$. *Ans.* $a^{\frac{3}{4}} - a^{\frac{1}{4}}$.

11. Multiply $2\sqrt[3]{x^2} + \sqrt{xy}$ by $3\sqrt[3]{x} - \sqrt{xy}$.
Ans. $6x + 3\sqrt[6]{x^5y^3} - 2\sqrt[6]{x^7y^3} - xy$.

12. Multiply $a^{\frac{1}{2}} - 2a^{-\frac{1}{4}} + a^{-1}$ by $a^{\frac{1}{4}} - a^{-\frac{1}{2}}$.
Ans. $a^{\frac{3}{4}} - 3 + 3a^{-\frac{3}{4}} - a^{-\frac{3}{2}}$.

13. Divide $a - b$ by $\sqrt{a} + \sqrt{b}$. *Ans.* $\sqrt{a} - \sqrt{b}$.

14. Divide $a^{\frac{5}{6}} - 2a^{\frac{1}{2}} + a^{\frac{1}{6}}$ by $a^{\frac{1}{3}} - 1$. *Ans.* $a^{\frac{1}{2}} - a^{\frac{1}{6}}$.

15. Multiply $a^{\frac{5}{2}} + a^2b^{\frac{2}{3}} + a^{\frac{3}{2}}b^{\frac{4}{3}} + ab^2 + a^{\frac{1}{2}}b^{\frac{8}{3}} + b^{\frac{10}{3}}$ by $a^{\frac{1}{2}} - b^{\frac{2}{3}}$.
Ans. $a^3 - b^4$.

16. Divide $x^{\frac{4}{3}} + x^{\frac{2}{3}}a^{\frac{2}{3}} + a^{\frac{4}{3}}$ by $x^{\frac{2}{3}} + x^{\frac{1}{3}}a^{\frac{1}{3}} + a^{\frac{2}{3}}$.
Ans. $x^{\frac{2}{3}} - x^{\frac{1}{3}}a^{\frac{1}{3}} + a^{\frac{2}{3}}$.

17. Simplify $\left(a^{\frac{1}{3}}a^{\frac{5}{7}}\right)^{\frac{7}{11}}$. *Ans.* $a^{\frac{2}{3}}$.

18. Simplify $\left(\frac{1 \pm \sqrt{5}}{2}\right)^3$. *Ans.* $2 \pm \sqrt{5}$.

19. Simplify $\dfrac{\left\{(a^m)^{\frac{1}{r}}(a^s)^{\frac{1}{n}}(c^{-\frac{m}{n}})\right\}^{nr}}{\left\{\sqrt[s]{c^n}\left(\sqrt[m]{c}\right)^r\left(\sqrt[s]{a}\right)^r\right\}^{ms}}$. *Ans.* $\left(\frac{a}{c}\right)^{mn+rs-mr}$.

20. Simplify $\left\{\dfrac{2\sqrt{3}\cdot 2\sqrt[3]{108}}{3\sqrt[3]{72}\cdot 3\,(3)^{\frac{1}{2}}}\right\}^{\frac{1}{2}}$. *Ans.* $\frac{2}{3}\sqrt[6]{\frac{3}{2}}$.

21. Simplify $\left\{\dfrac{\frac{4}{5}\sqrt[3]{\frac{1}{5}} - 2\sqrt{\frac{1}{2}}}{\frac{1}{2}\sqrt[3]{5} + \frac{5}{4}\sqrt[4]{2}}\right\}^{\frac{1}{2}}$ *Ans.* $\frac{2}{5}\left(2\sqrt[3]{5} - 5\sqrt[4]{2}\right)^{\frac{1}{2}}$.

22. Simplify $\dfrac{(\sqrt{a} + \sqrt{b})(\sqrt{a} - \sqrt{b})}{(\sqrt[4]{a} + \sqrt[4]{b})(\sqrt[4]{a} - \sqrt[4]{b})}$. *Ans.* $\sqrt{a} + \sqrt{b}$.

23. Simplify $\left\{\dfrac{(\sqrt{5} + 2)(\sqrt[4]{5} + \sqrt{2})(\sqrt[4]{5} - \sqrt{2})}{(\sqrt{13} + 3)(\sqrt[4]{13} + \sqrt{3})(\sqrt[4]{13} - \sqrt{3})}\right\}^{\frac{1}{2}}$.
Ans. $\frac{1}{2}$.

IMAGINARY QUANTITIES

262. It has been shown (**228**, 3) that an even root of a negative quantity is *imaginary*, an expression for such a root being a symbol of an impossible operation. Thus if we take a^2, which is numerically a perfect square, and affect it with the minus sign, we cannot obtain the square root of the result. For,

$$(+a)^2 = +a^2,$$
$$(-a)^2 = +a^2.$$

Hence the indicated root, $\sqrt{-a^2}$ is not *real*, but *imaginary*. Such expressions are, however, of frequent occurrence in analysis and its application to physical science, and conclusions of the highest importance depend upon their use and proper interpretation. We therefore proceed to investigate the rules to be observed in operating with such quantities.

263. When a real and an imaginary quantity are connected in a single expression, the whole is considered imaginary on account of the presence of the imaginary part. Thus the binomial, $4 + \sqrt{-3}$, considered as a single quantity, is imaginary.

264. According to **227**,

$$\sqrt{-a} = \sqrt{a \times (-1)} = \sqrt{a} \cdot \sqrt{-1};$$

also, $\sqrt{-a^2 - b^2 + 2ab} = \sqrt{(a-b)^2 \times (-1)} = (a-b)\sqrt{-1}$.
Hence, if we regard only quadratic expressions, every imaginary quantity may be reduced to the form,

$$a \pm b\sqrt{-1},$$

in which a is the *real* part, b the coefficient of the imaginary part, and $\sqrt{-1}$ the *imaginary factor*. Thus we may employ only the single symbol, $\sqrt{-1}$, to indicate that a quantity is imaginary.

265. For convenience in multiplication and division of imaginary quantities, we will now obtain some of the successive powers of the symbol $\sqrt{-1}$, and deduce the law of their formation.

$$\begin{aligned}
(\sqrt{-1})^1 &&&= +\sqrt{-1},\\
(\sqrt{-1})^2 &= &(\sqrt{-1}) \times (\sqrt{-1}) &= -1,\\
(\sqrt{-1})^3 &= &(-1) \times (\sqrt{-1}) &= -\sqrt{-1},\\
(\sqrt{-1})^4 &= &(-\sqrt{-1}) \times (\sqrt{-1}) &= +1
\end{aligned}$$

Multiplying these powers, in their order, by the 4th, we shall obtain the 5th, 6th, 7th, and 8th, the same as the 1st, 2d, 3d, and 4th; and so on.

266. The common rules for multiplication and division of radicals will apply to imaginary quantities, *with a simple modification respecting the law of signs.*

Let it be required to find the product of $\sqrt{-a}$ and $\sqrt{-b}$.

To obtain the true result, we must separate the imaginary symbol $\sqrt{-1}$ from each factor. Thus,

$$\begin{aligned}\sqrt{-a} \times \sqrt{-b} &= \sqrt{a} \cdot \sqrt{-1} \times \sqrt{b} \cdot \sqrt{-1} \\ &= \sqrt{ab} \times (\sqrt{-1})^2 \\ &= \sqrt{ab} \times (-1) \qquad \text{[From } \mathbf{265}\text{]}, \\ &= -\sqrt{ab},\end{aligned}$$

a *real* quantity, and *negative.*

But if we multiply by the common rule for radicals (**253**), we shall have

$$\sqrt{-a} \times \sqrt{-b} = \sqrt{(-a) \cdot (-b)} = \sqrt{ab},$$

a result erroneous with respect to the sign before the radical.

Proceeding as in the first operation, we find that

$$(-\sqrt{-a}) \times (-\sqrt{-b}) = +\sqrt{ab} \cdot (-1) = -\sqrt{ab};$$

$$(+\sqrt{-a}) \times (-\sqrt{-b}) = -\sqrt{ab} \cdot (-1) = +\sqrt{ab}.$$

Thus, like signs produce $-$, and unlike signs produce $+$. Hence,

1. *The product of two imaginary terms will be real, and the sign before the radical will be determined by the common rule reversed.*

We may operate in like manner in division of imaginary quantities. Thus,

$$\frac{+\sqrt{-ab}}{+\sqrt{-a}} = \frac{+\sqrt{ab} \cdot \sqrt{-1}}{+\sqrt{a} \cdot \sqrt{-1}} = +\sqrt{b};$$

$$\frac{-\sqrt{-ab}}{+\sqrt{-a}} = \frac{-\sqrt{ab} \cdot \sqrt{-1}}{+\sqrt{a} \cdot \sqrt{-1}} = -\sqrt{b}.$$

That is, like signs produce $+$ and unlike signs produce $-$. Hence,

2. *The quotient of one imaginary term divided by another will be real, and the sign before the radical will be governed by the common rule.*

267. Let us assume the equation

$$a + b\sqrt{-1} = a' + b'\sqrt{-1} \quad . \quad . \quad . \quad (1),$$

in which a and a' are real. By transposition,

$$a - a' = (b' - b)\sqrt{-1} \quad . \quad . \quad . \quad (2).$$

Now it is evident that in this equation

$$a = a'.$$

For, if $a > a'$, or $a < a'$, then the first member of equation (2) is *different from zero,* and *real.* But this cannot be, because the second member is either *nothing* or *imaginary.* Hence $a = a'$; and equation (2) becomes

$$0 = (b' - b)\sqrt{-1},$$

which can only be satisfied by putting

$$b = b'.$$

Hence,

If two imaginary quantities are equal, then the real parts are equal, and the coefficients of the imaginary symbol are also equal.

268. These principles may now be applied in the following

EXAMPLES.

1. Multiply $a\sqrt{-c}$ by $b\sqrt{-d}$. *Ans.* $-ab\sqrt{cd}$.
2. Multiply $2\sqrt{-6}$ by $\sqrt{-15}$. *Ans.* $-6\sqrt{10}$.
3. Multiply $-\sqrt{-ac}$ by $\sqrt{-ad}$. *Ans.* $a\sqrt{cd}$.
4. Multiply $3\sqrt{-2}$ by $\sqrt{5}$. *Ans.* $3\sqrt{-10}$.
5. Multiply $3 + \sqrt{-5}$ by $7 - \sqrt{-5}$. *Ans.* $26 + 4\sqrt{-5}$.
6. Multiply $\sqrt{a} + \sqrt{-c}$ by $\sqrt{-a} + \sqrt{c}$. *Ans.* $(a+c)\sqrt{-1}$.
7. Divide $9\sqrt{-10}$ by $3\sqrt{-2}$. *Ans.* $3\sqrt{5}$.
8. Divide $a\sqrt{-b}$ by $c\sqrt{-d}$. *Ans.* $\dfrac{a}{c}\sqrt{\dfrac{b}{d}}$.

9. Reduce $\dfrac{\sqrt{-1}}{2\sqrt{-3}}$ to simpler terms. *Ans.* $\frac{1}{6}\sqrt{3}$.

10. Expand $(a+\sqrt{-c})^4$.
Ans. $a^4-6a^2c+c^2+(4a^3-4ac)\sqrt{-c}$.

11. Divide $a^2+\sqrt{-a}$ by $a-\sqrt{-a}$. *Ans.* $a+\sqrt{-a}-1$.

12. Find the values of x and y in the equation $a+y+x\sqrt{-c}=c+x+y\sqrt{-a}$.
Ans. $\begin{cases} x=a+\sqrt{ac}\,; \\ y=c+\sqrt{ac}. \end{cases}$

PROPERTIES OF QUADRATIC SURDS.

269. A ***Quadratic Surd*** is the square root of an imperfect square.

270. A radical will be a surd, if it contains an irrational factor when reduced to its simplest form. Thus $\sqrt{12}$ is a surd, for $\sqrt{12}=2\sqrt{3}$. The surd factor $\sqrt{3}$, is called the *irrational part* of the given surd.

271. A quantity may be a surd when considered *algebraically*, even though its *numerical* value is rational. Thus, the quantity, $\sqrt{a+2b}$ is a surd, considered as an algebraic expression. But if $a=13$ and $b=6$, we have $\sqrt{a+2b}=\sqrt{13+12}=\sqrt{25}=5$, a rational quantity.

272. The following properties of surds are important both in a theoretical and a practical view. The radical expressions are supposed to represent irrational numbers.

1. *The product of two quadratic surds which have not the same irrational part, is irrational.*

Let $a\sqrt{b}$ and $c\sqrt{d}$ be the two surds, reduced to their simplest form. Their product will be

$$ac\sqrt{bd}.$$

And since, by hypothesis, b and d are not the same numbers, one of them must contain at least a factor which the other does not. But this factor must be irrational, otherwise the given

surds are not in their simplest form. Therefore the product $ac\sqrt{bd}$ is irrational (**270**).

2. *The sum or difference of two quadratic surds which have not the same irrational part, cannot be equal to a rational quantity.*

Let $\sqrt{a}$ and $\sqrt{b}$ be the two surds; and, if possible, suppose

$$\sqrt{a} + \sqrt{b} = c \quad . \quad . \quad . \quad . \quad (1),$$

c being rational. Squaring both members, and transposing $a + b$,

$$2\sqrt{ab} = c^2 - a - b \quad . \quad . \quad . \quad (2).$$

That is, we have an irrational quantity equal to a rational quantity, which is impossible. Therefore equation (1) cannot be true.

In like manner it can be shown that the difference of two surds, not having the same irrational part, cannot be rational.

3. *The sum or difference of two quadratic surds which have not the same irrational part, cannot be equal to another quadratic surd.*

If possible, suppose $\sqrt{a} + \sqrt{b} = \sqrt{c}$, in which c is rational, but $\sqrt{c}$ a surd.

Squaring both members, and transposing,

$$2\sqrt{ab} = c - a - b,$$

which is impossible, because a surd cannot be equal to a rational quantity.

4. *In any equation which involves both rational quantities and quadratic surds, the rational parts of the members are equal, and also the irrational parts.*

Suppose

$$a + b\sqrt{x} = c + d\sqrt{y} \quad . \quad . \quad . \quad (1),$$

the surds being in their simplest form. By transposition,

$$b\sqrt{x} - d\sqrt{y} = c - a \quad . \quad . \quad . \quad (2).$$

Since the second member is rational, equation (2) cannot be true if the surds have not the same irrational part (2). Therefore $\sqrt{x} = \sqrt{y}$, and the equation may be written,

$$(b - d)\sqrt{x} = c - a \quad . \quad . \quad . \quad (3),$$

which can be true only when $b - d = 0$ and $c - a = 0$; for otherwise, we should have a surd equal to a rational quantity. Hence, in (1), $a = c$, and $b\sqrt{x} = d\sqrt{y}$.

SQUARE ROOT OF A BINOMIAL SURD.

273. A ***Binomial Surd*** is a binomial, one or both of whose terms are surds. Thus, $3+\sqrt{5}$ and $\sqrt{7}-\sqrt{2}$ are binomial surds.

274. If we square a binomial surd in the form of $a \pm \sqrt{b}$ or $\sqrt{a} \pm \sqrt{b}$, the result will be a binomial surd. Thus,

$$(3+\sqrt{5})^2 = 14 + 6\sqrt{5};$$
$$(\sqrt{7}-\sqrt{2})^2 = 9 - 2\sqrt{14}.$$

Hence, *a binomial surd in the form of* $a \pm \sqrt{b}$ *may sometimes be a perfect square.*

275. To obtain a rule for extracting the square root of a binomial surd in the form of $a \pm \sqrt{b}$, let us assume

$$\sqrt{x}+\sqrt{y} = \sqrt{a+\sqrt{b}} \quad \ldots \quad (1),$$

in which one or both of the terms in the first member must be irrational, because the second member is a surd.

Squaring both members,

$$x + 2\sqrt{xy} + y = a + \sqrt{b} \quad \ldots \quad (2).$$

Hence, from **272**, 4, $\quad x + y = a \quad \ldots \quad (3).$

$$2\sqrt{xy} = \sqrt{b} \quad \ldots \quad (4).$$

Subtracting (4) from (3), and then taking the square root of the result,

$$\sqrt{x}-\sqrt{y} = \sqrt{a-\sqrt{b}} \quad \ldots \quad (5).$$

Multiplying (1) by (5),

$$x - y = \sqrt{a^2-b} \quad \ldots \quad (6).$$

Combining (3) and (6), we obtain

$$x = \frac{a+\sqrt{a^2-b}}{2} \quad \ldots \quad (7),$$

$$y = \frac{a-\sqrt{a^2-b}}{2} \quad \ldots \quad (8).$$

Now it is obvious from these equations that x and y will be rational when a^2-b is a perfect square. Moreover, the values of x and y in (7) and (8) will evidently satisfy equations (1) and (5). Hence, to obtain the square root of a binomial surd, we may proceed as follows:

Let a represent the rational part, and $\sqrt{b}$ the radical part, and find the values of x and y in equations (7) and (8). Then if the binomial is in the form of $a+\sqrt{b}$, as in equation (1), the required root will be

$$\sqrt{x}+\sqrt{y}.$$

But if the binomial is in the form of $a-\sqrt{b}$, as in equation (5), the required root will be

$$\sqrt{x}-\sqrt{y}.$$

EXAMPLES FOR PRACTICE.

1. Required the square root of $7+4\sqrt{3}$.

In this example, $a=7$, and $\sqrt{b}=4\sqrt{3}$; or $b=48$. Hence,

$$x=\frac{7+\sqrt{49-48}}{2}=4;$$

$$y=\frac{7-\sqrt{49-48}}{2}=3.$$

And we have $\qquad \sqrt{x}+\sqrt{y}=2+\sqrt{3}$, *Ans.*

2. Required the square root of $11-8\sqrt{-5}$.

In this example $a=11$ and $\sqrt{b}=8\sqrt{-5}$, or $b=-320$. We have, therefore,

$$x=\frac{11+\sqrt{121+320}}{2}=16;$$

$$y=\frac{11-\sqrt{121+320}}{2}=-5.$$

Hence, $\qquad \sqrt{x}-\sqrt{y}=4-\sqrt{-5}$, *Ans.*

3. Required the square root of $5m^2-c+4m\sqrt{m^2-c}$.

We have $a=5m^2-c$, and $b=16m^4-16m^2c$. Whence,

$$x=\frac{5m^2-c+\sqrt{(5m^2-c)^2-(16m^4-16m^2c)}}{2}=4m^2;$$

$$y=\frac{5m^2-c-\sqrt{(5m^2-c)^2-(16m^4-16m^2c)}}{2}=m^2-c;$$

and we have $\qquad \sqrt{x}+\sqrt{y}=2m+\sqrt{m^2-c}$, *Ans.*

4. What is the square root of $11 + 6\sqrt{2}$? *Ans.* $3 + \sqrt{2}$.

5. What is the square root of $7 - 4\sqrt{3}$? *Ans.* $2 - \sqrt{3}$.

6. What is the square root of $7 - 2\sqrt{10}$? *Ans.* $\sqrt{5} - \sqrt{2}$.

7. What is the square root of $94 + 42\sqrt{5}$? *Ans.* $7 + 3\sqrt{5}$.

8. What is the square root of $28 + 10\sqrt{3}$? *Ans.* $5 + \sqrt{3}$.

9. What is the square root of $np + 2m^2 - 2m\sqrt{np + m^2}$?
Ans. $\sqrt{np + m^2} - m$.

10. What is the square root of $bc + 2b\sqrt{bc - b^2}$?
Ans. $b + \sqrt{bc - b^2}$.

11. What is the square root of $7 + 30\sqrt{-2}$?
Ans. $5 + 3\sqrt{-2}$.

12. Find the value of $\sqrt{16 + 30\sqrt{-1}} + \sqrt{16 - 30\sqrt{-1}}$.
Ans. 10.

13. Find the value of $\sqrt{11 + 6\sqrt{2}} + \sqrt{7 - 2\sqrt{10}}$.
Ans. $3 + \sqrt{5}$.

14. Find the value of $\sqrt{31 + 12\sqrt{-5}} + \sqrt{-1 + 4\sqrt{-5}}$.
Ans. $8 + 2\sqrt{-5}$.

15. Find the value of $\sqrt[4]{17 + 12\sqrt{2}}$, *Ans.* $1 + \sqrt{2}$.

RATIONALIZATION.

276. It is sometimes useful to transform a fraction whose denominator is a surd, in such a manner that the denominator shall become *rational.* The fraction is thus simplified, because, in general, its numerical value can be more readily calculated. This transformation is usually effected by multiplying both terms of the fraction by the same factor.

277. The process of clearing a quantity of radical signs by multiplication is called *Rationalization*, and we will now consider how we may find the factor which will rationalize a surd quantity, in some of the more important cases.

278. To find a factor which will rationalize any monomial.

It is evident that the factor in this case will be the monomial itself, with an index equal to the difference between unity and the given index. For, we have in general terms,

$$a^{\frac{1}{n}} \times a^{1-\frac{1}{n}} = a.$$

1. Rationalize $\sqrt{a}$.

The factor is $\sqrt{a}$; for, $\sqrt{a} \times \sqrt{a} = a$, a rational quantity.

2. Rationalize $x^{\frac{3}{5}}$.

The factor is $x^{\frac{2}{5}}$; for, $x^{\frac{3}{5}} \times x^{\frac{3}{5}} = x^{\frac{5}{5}} = x$.

279. To find a factor which will rationalize a binomial in the form of $a \pm \sqrt[m]{b}$, or $\sqrt[m]{a} \pm \sqrt[n]{b}$, m and n being each some power of 2.

The product of the sum and difference of two quantities is equal to the difference of their squares. Hence, if we multiply the given binomial in this case, by the same terms connected by the opposite sign, the indices of the product must be respectively the halves of the given indices; and a repetition of the process will ultimately rationalize the quantity, provided m and n are any powers of 2.

1. Rationalize $a + \sqrt{c}$.

The factor is $a - \sqrt{c}$; for we have

$$(a + \sqrt{c})(a - \sqrt{c}) = a^2 - c.$$

2. Rationalize $\sqrt{a} - \sqrt[4]{x}$.

$$(\sqrt{a} - \sqrt[4]{x})(\sqrt{a} + \sqrt[4]{x}) = a - \sqrt{x};$$

$$(a - \sqrt{x})(a + \sqrt{x}) = a^2 - x,$$

a rational result; and the complete multiplier is

$$(\sqrt{a} + \sqrt[4]{x})(a + \sqrt{x}).$$

A *trinomial* may be treated in a similar manner, when it contains only radicals of the 2d degree.

3. Rationalize $\sqrt{5}+\sqrt{2}-\sqrt{3}$.

$$\begin{array}{l}
\sqrt{5}+\sqrt{2}-\sqrt{3} \\
\underline{\sqrt{5}+\sqrt{2}+\sqrt{3}} \\
5+\sqrt{10}-\sqrt{15} \\
\quad\quad \sqrt{10}+2-\sqrt{6} \\
\quad\quad\quad\quad \underline{\sqrt{15}+\sqrt{6}-3} \\
4+2\sqrt{10} \\
\underline{4-2\sqrt{10}} \\
16-40=-24,
\end{array}$$

a rational result; and the complete multiplier is

$$(\sqrt{5}+\sqrt{2}+\sqrt{3})(4-2\sqrt{10}).$$

Thus we perceive that it is necessary simply to change the sign of one of the terms of the trinomial, and multiply by the result, repeating the process with the product.

280. To find a factor which will rationalize any binomial surd whatever.

Let the binomial be represented by the general form,

$$a^{\frac{1}{r}} \pm b^{\frac{1}{s}}.$$

Put $x=a^{\frac{1}{r}}$, and $y=b^{\frac{1}{s}}$; and let n be the *least common multiple* of r and s. Then $x^n \pm y^n$ will be rational. But $x+y$ will exactly divide x^n+y^n when n is *odd*, and x^n-y^n when n is *even;* and $x-y$ will exactly divide x^n-y^n whether n is odd or even (89). These quotients will therefore be the factors that will rationalize the respective divisors. Hence, let q represent the required factor; then

(1) $\quad q=\dfrac{x^n+y^n}{x+y}$, for $a^{\frac{1}{r}}+b^{\frac{1}{s}}$, when n is odd;

(2) $\quad q=\dfrac{x^n-y^n}{x+y}$, for $a^{\frac{1}{r}}+b^{\frac{1}{s}}$, when n is even;

(3) $\quad q=\dfrac{x^n-y^n}{x-y}$, for $a^{\frac{1}{r}}-b^{\frac{1}{s}}$, n being odd or even.

1. Rationalize the binomial $a^{\frac{1}{2}} + b^{\frac{1}{3}}$.

Since $n = 6$, an even number, we have from (2),

$$q = \frac{x^6 - y^6}{x + y} = \frac{a^3 - b^2}{a^{\frac{1}{2}} + b^{\frac{1}{3}}} = a^{\frac{5}{2}} - a^2b^{\frac{1}{3}} + a^{\frac{3}{2}}b^{\frac{2}{3}} - ab + a^{\frac{1}{2}}b^{\frac{4}{3}} - b^{\frac{5}{3}},$$

the factor required; and

$$\left(a^{\frac{1}{2}} + b^{\frac{1}{3}}\right) \times \left(a^{\frac{5}{2}} - a^2b^{\frac{1}{3}} + a^{\frac{3}{2}}b^{\frac{2}{3}} - ab + a^{\frac{1}{2}}b^{\frac{4}{3}} - b^{\frac{5}{3}}\right) = a^3 - b^2,$$

the rational result.

The foregoing methods may be applied in the solution of the following

EXAMPLES.

1. Reduce $\frac{a}{\sqrt{c}}$ to a fraction whose denominator shall be rational.

Ans. $\frac{a\sqrt{c}}{c}$.

2. Reduce $\frac{\sqrt{10} - \sqrt{15}}{\sqrt{6}}$ to a fraction whose denominator shall be rational.

Ans. $\frac{2\sqrt{15} - 3\sqrt{10}}{6}$.

3. Reduce $\frac{x}{\sqrt[3]{a}}$ to a fraction whose denominator shall be rational.

Ans. $\frac{x\sqrt[3]{a^2}}{a}$.

4. Reduce $\frac{\sqrt{2}}{\sqrt[3]{9}}$ to a fraction whose denominator shall be rational.

Ans. $\frac{\sqrt[6]{72}}{3}$.

5. Reduce $\frac{5}{\sqrt{7} + \sqrt{3}}$ to a fraction whose denominator shall be rational.

Ans. $\frac{5\left(\sqrt{7} - \sqrt{3}\right)}{4}$.

6. Reduce $\frac{\sqrt{a}}{\sqrt{a} - \sqrt{c}}$ to a fraction whose denominator shall be rational.

Ans. $\frac{a + \sqrt{ac}}{a - c}$.

7. Reduce $\dfrac{\sqrt{11}+\sqrt{5}}{\sqrt{11}-\sqrt{5}}$ to a rational denominator.

Ans. $\dfrac{8+\sqrt{55}}{3}$.

8. Reduce $\dfrac{8}{\sqrt{11}+\sqrt{3}}$ to its simplest form. *Ans.* $\sqrt{11}-\sqrt{3}$.

9. Reduce $\dfrac{\sqrt{10}+\sqrt{6}}{\sqrt{10}-\sqrt{6}}$ to its simplest form. *Ans.* $4+\sqrt{15}$.

10. Reduce $\dfrac{\sqrt{5}-\sqrt{-3}}{\sqrt{5}+\sqrt{-3}}$ to its simplest form. *Ans.* $\dfrac{1-\sqrt{-15}}{4}$.

11. Simplify $\dfrac{(3+\sqrt{3})(3+\sqrt{5})(\sqrt{5}-2)}{(5-\sqrt{5})(\sqrt{3}+1)}$. *Ans.* $\frac{1}{5}\sqrt{15}$.

12. Simplify $\dfrac{1+a+\sqrt{1-a^2}}{1+a-\sqrt{1-a^2}}$. *Ans.* $\dfrac{1+\sqrt{1-a^2}}{a}$.

13. Find the factor which will rationalize $\sqrt{5}-\sqrt[4]{2}$.

Ans. $5\sqrt{5}+\sqrt{10}+5\sqrt[4]{2}+\sqrt[4]{8}$.

14. Reduce $\dfrac{a^{\frac{1}{2}}b^{\frac{2}{3}}}{a^{\frac{1}{2}}-b^{\frac{2}{3}}}$ to its simplest form.

Ans. $\dfrac{a^3b^{\frac{2}{3}}+a^{\frac{5}{2}}b^{\frac{4}{3}}+a^2b^2+a^{\frac{3}{2}}b^{\frac{8}{3}}+ab^{\frac{10}{3}}+a^{\frac{1}{2}}b^4}{a^3-b^4}$.

RADICAL EQUATIONS.

281. A ***Radical Equation*** is one in which the unknown quantity is affected by the radical sign.

282. In order to solve a radical equation, it is necessary in the first place to rationalize the terms containing the unknown quantity. In case of fractional terms, this may be effected in part by methods already explained. But the process is commonly one of involution.

The following are examples of *simple equations* containing radical quantities.

1. Given $\sqrt{x+11}+\sqrt{x-4}=5$ to find x.

OPERATION.

$$\sqrt{x+11}+\sqrt{x-4}=5;$$

by transposition, $\sqrt{x+11}=5-\sqrt{x-4}$;
squaring and reducing, $x+11=x+21-10\sqrt{x-4}$;
transposing and reducing, $\sqrt{x-4}=1$;
squaring and reducing, $x=5$.

2. Given $\dfrac{\sqrt{x}-\sqrt{x-5}}{\sqrt{x}+\sqrt{x-5}}=\dfrac{4x-35}{5}$ to find x.

OPERATION.

$$\frac{\sqrt{x}-\sqrt{x-5}}{\sqrt{x}+\sqrt{x-5}}=\frac{4x-35}{5}.$$

Multiplying both terms of the fraction by the numerator (**279**), $\dfrac{2x-5-2\sqrt{x^2-5x}}{5}=\dfrac{4x-35}{5}$;
clearing of fractions, etc., $-2\sqrt{x^2-5x}=2x-30$;
dividing by 2, $-\sqrt{x^2-5x}=x-15$;
squaring and reducing, $25x=225$;
whence, $x=9$.

3. Given $\dfrac{c}{\sqrt{x}+\sqrt{a}}+\dfrac{m\sqrt{a}}{x-a}=\dfrac{m}{\sqrt{x}-\sqrt{a}}$ to find x.

The least common multiple of the denominators is $x-a=(\sqrt{x}+\sqrt{a})(\sqrt{x}-\sqrt{a})$; and the solution will be as in the following

OPERATION.

$$\frac{c}{\sqrt{x}+\sqrt{a}}+\frac{m\sqrt{a}}{x-a}=\frac{m}{\sqrt{x}-\sqrt{a}},$$
$$c(\sqrt{x}-\sqrt{a})+m\sqrt{a}=m(\sqrt{x}+\sqrt{a}),$$
$$(c-m)\sqrt{x}=c\sqrt{a},$$
$$\sqrt{x}=\frac{c\sqrt{a}}{c-m},$$
$$x=\frac{ac^2}{(c-m)^2},\ \textit{Ans.}$$

From these illustrations, we derive the following precepts for the solution of radical equations:

1. It is sometimes advantageous to rationalize the denominator of a fractional term, before transposition or involution.

2. An equation should be simplified as much as possible before involution; and care should be taken so to dispose the terms in the two members as to secure the simplest results after involution.

EXAMPLES FOR PRACTICE.

Find the values of the unknown quantity in each of the following equations:

1. $\sqrt{x+7}+\sqrt{x}=7.$ *Ans.* $x=9.$

2. $x+3=\sqrt{x^2-4x+59}.$ *Ans.* $x=5.$

3. $\sqrt{\sqrt{x+48}-\sqrt{x}}=\sqrt[4]{x}.$ *Ans.* $x=16.$

4. $\sqrt[6]{x+2\sqrt{a+x}}=\sqrt[3]{a-\sqrt{a+x}}.$ *Ans.* $x=\dfrac{a^2-4a}{4}.$

5. $\dfrac{a}{\sqrt{x}}+\dfrac{\sqrt{x}}{c}=\sqrt{\dfrac{n}{x}}.$ *Ans.* $x=c(\sqrt{n}-a).$

6. $\dfrac{\sqrt{1-x}}{\sqrt{1+x}}+\dfrac{1+x}{\sqrt{1-x^2}}=\dfrac{3x}{\sqrt{1-x^2}}.$ *Ans.* $x=\dfrac{2}{3}.$

7. $\sqrt{c+x}=\dfrac{\sqrt{a+x^2}}{\sqrt{c+x}}.$ *Ans.* $x=\dfrac{a-c^2}{2c}.$

8. $x-\sqrt{9+x\sqrt{x^2-3}}=3.$ *Ans.* $x=3\frac{1}{4}.$

9. $2\sqrt{x}-2\sqrt{x-32}=\sqrt{32}.$ *Ans.* $x=50.$

10. $\dfrac{a}{\sqrt{x}-2}-\dfrac{a+c}{x-4}=\dfrac{c}{\sqrt{x}+2}.$ *Ans.* $x=\left(-\dfrac{a+c}{a-c}\right)^2.$

11. $\dfrac{\sqrt{mx}-\sqrt{m}}{\sqrt{cx}-\sqrt{c}}=\dfrac{\sqrt{x}+m}{\sqrt{x}+c}.$ *Ans.* $x=mc.$

12. $\sqrt[3]{a^3-3a^2x+x^2\sqrt{3a-x}}=a-x.$ *Ans.* $x=3a-1.$

13. $x + \sqrt{c^2 - ax} = \dfrac{c^2}{\sqrt{c^2 - ax}}$. *Ans.* $x = \dfrac{c^2 - a^2}{a}$.

14. $\dfrac{1}{x} + \dfrac{1}{5} = \sqrt{\dfrac{1}{25} + \dfrac{1}{x}\sqrt{\dfrac{1}{5} + \dfrac{1}{x^2}}}$. *Ans.* $x = 20$.

15. $\sqrt{a - x} = \sqrt[4]{a + x^2}$. *Ans.* $x = \dfrac{a-1}{2}$.

16. $\dfrac{\sqrt[4]{1+x}}{\sqrt{2 - \sqrt{x}}} = \dfrac{\sqrt{2 + \sqrt{x}}}{\sqrt[4]{4 + x}}$. *Ans.* $x = \dfrac{12}{13}$.

17. $\sqrt[3]{5 + x} + \sqrt[3]{5 - x} = \sqrt[3]{10}$. *Ans.* $x = 5$.

18. $\sqrt{x} + \sqrt{a + x} = \dfrac{2a}{\sqrt{a + x}}$. *Ans.* $x = \dfrac{a}{3}$.

19. $x + a = \sqrt{a^2 + x\sqrt{b^2 + x^2}}$. *Ans.* $x = \dfrac{b^2 - 4a^2}{4a}$.

20. $\dfrac{\sqrt{6x} - 2}{\sqrt{6x} + 2} = \dfrac{4\sqrt{6x} - 9}{4\sqrt{6x} + 6}$. *Ans.* $x = 6$.

21. $\sqrt[3]{64 + x^2 - 8x} = \dfrac{4 + x}{\sqrt[3]{4 + x}}$. *Ans.* $x = 3$.

22. $\sqrt{5 + x} + \sqrt{x} = \dfrac{15}{\sqrt{5 + x}}$. *Ans.* $x = 4$.

23. $\sqrt{x + \sqrt{x}} - \sqrt{x - \sqrt{x}} = \dfrac{3}{2}\left(\dfrac{x}{x + \sqrt{x}}\right)^{\frac{1}{2}}$. *Ans.* $x = \dfrac{25}{16}$.

24. $\dfrac{\sqrt{ax} - b}{\sqrt{ax} + b} = \dfrac{3\sqrt{ax} - 2b}{3\sqrt{ax} + 5b}$. *Ans.* $x = \dfrac{9b^2}{a}$.

25. $\dfrac{\sqrt{4x + 1} + \sqrt{4x}}{\sqrt{4x + 1} - \sqrt{4x}} = 9$. *Ans.* $x = \dfrac{4}{9}$.

26. $\dfrac{3\sqrt{x} - 4}{\sqrt{x} + 2} = \dfrac{3\sqrt{x} + 15}{\sqrt{x} + 40}$. *Ans.* $x = 4$.

27. $\dfrac{\sqrt{x} + \sqrt{x - a}}{\sqrt{x} - \sqrt{x - a}} = \dfrac{n^2 a}{x - a}$. *Ans.* $x = a\dfrac{(1 \pm n)^2}{1 \pm 2n}$.

SECTION V.

QUADRATIC EQUATIONS.

283. A ***Quadratic Equation*** is an equation of the second degree, or one which contains the second power of the unknown quantity, and no higher power; as $3x^2 = 48$, and $ax^2 - 2bx = c$. That term of the equation which does not contain the unknown quantity, is called the *absolute term.*

284. Quadratic equations are divided into two classes—Pure Quadratics, and Affected Quadratics.

PURE QUADRATICS.

285. A ***Pure Quadratic Equation*** is one which contains the second power only, of the unknown quantity; as $3x^2 - 7 = 20$.

NOTE.—A *pure equation,* in general, is one which contains but a single power of the unknown quantity.

286. It is evident that by the rule for solving simple equations, every pure quadratic may be reduced to the form of

$$x^2 = a,$$

in which a may be any quantity, real or imaginary, positive or negative.

Extracting the square root of both members of this equation, we have

$$x = +\sqrt{a} \quad \text{or} \quad -\sqrt{a}.$$

Hence,

Every pure quadratic equation has two roots, equal in numerical value, but of opposite signs.

1. Given $\dfrac{x^2 - 4}{6} - \dfrac{x^2 - 24}{4} = \dfrac{x^2}{2} - 32$ to find the values of x.

OPERATION.

$$\frac{x^2-4}{6} - \frac{x^2-24}{4} = \frac{x^2}{2} - 32;$$

clearing of fractions, $2x^2 - 8 - 3x^2 + 72 = 6x^2 - 384;$
collecting terms, $7x^2 = 448;$
dividing by 7, $x^2 = 64;$
by evolution, $x = \pm 8$, *Ans.*

We have, therefore, for the solution of pure quadratics, the following

RULE.—*Reduce the equation to the form of* $x^2 = a$, *and then take the square root of both members.*

EXAMPLES FOR PRACTICE.

Find the values of x in each of the following equations:

1. $3x^2 - 16 = x^2 + 2$. *Ans.* $x = \pm 3$.

2. $2x^2 - 54 = 126 - 3x^2$. *Ans.* $x = \pm 6$.

3. $7x^2 + 8 = 57 + 3x^2 + 15$. *Ans.* $x = \pm 4$.

4. $bx^2 + 2d = 2cx^2 + a$. *Ans.* $x = \pm\sqrt{\dfrac{a-2d}{b-2c}}$.

5. $ax^2 + 1 = (a-x)(a+x)$. *Ans.* $x = \pm\sqrt{a-1}$.

6. $\dfrac{x+4}{x-4} + \dfrac{x-4}{x+4} = \dfrac{10}{3}$. *Ans.* $x = \pm 8$.

7. $\dfrac{x+2}{x-2} + \dfrac{x-2}{x+2} = \dfrac{13}{6}$. *Ans.* $x = \pm 10$.

8. $\dfrac{x+a}{x-a} + \dfrac{x-a}{x+a} = 7$. *Ans.* $x = \pm\dfrac{3a}{\sqrt{5}}$.

9. $\dfrac{x}{4} + \dfrac{4}{x} = \dfrac{x}{3} + \dfrac{3}{x}$. *Ans.* $x = \pm\sqrt{12}$.

10. $\dfrac{x}{a} + \dfrac{a}{x} = \dfrac{x}{c} + \dfrac{c}{x}$. *Ans.* $x = \pm\sqrt{ac}$.

11. $\dfrac{x^2-8}{6} = 1 + \sqrt{5}$. *Ans.* $x = \pm(3 + \sqrt{5})$.

12. $\dfrac{x^2-2\sqrt{2}}{3}-\dfrac{x^2-3}{2}=1-\sqrt{2}.$ *Ans.* $x=\pm(1+\sqrt{2})$.

13. $x^2+2x=9+\dfrac{18}{x}.$ *Ans.* $x=\pm 3$.

14. $\dfrac{3x^2}{2}-\dfrac{2x^2}{3}=1.$ *Ans.* $x=\pm 1.095445+$.

15. $\dfrac{x-4}{12}-\dfrac{(x-5)(x+5)}{x+4}=x-4.$

Ans. $x=\pm 4.54924+$.

AFFECTED QUADRATICS.

287. An ***Affected Quadratic Equation*** is one which contains both the first and the second powers of the unknown quantity; as $2x^2-3x=12$.

NOTES.—1. The two classes of quadratics, pure and affected, are sometimes called, respectively, *incomplete* and *complete* equations of the second degree.

2. A complete equation, in general, is one which contains every power of the unknown quantity, from the first to the highest inclusive. Thus a complete equation of the third degree must contain the first, second, and third powers of the unknown quantity.

288. Every affected quadratic equation may be reduced to the general form,

$$x^2+2ax=b,$$

in which $2a$ and b are positive or negative, integral or fractional.

For, to effect this, we have only to bring all the terms containing the unknown quantity into the first member, and all the known terms into the second member, and divide the result by the coefficient of x^2.

289. To solve a quadratic, suppose it first reduced to the form,

$$x^2+2ax=b.$$

To both members add a^2, or the square of one half the coefficient of x; thus

$$x^2+2ax+a^2=a^2+b.$$

The first member is now a *complete square.* Taking the square root of both members, we have

$$x + a = \pm \sqrt{a^2 + b};$$

whence by transposition,

$$x = -a \pm \sqrt{a^2 + b}.$$

Thus the equation has *two roots,* which are unequal in all cases except when $a^2 + b = 0$; in this case we shall have

$$x = a \pm 0,$$

and the equation is said to have *two equal roots.* Thus take the equation,

$$x^2 - 10x = -25.$$

Adding 25 to both members,

$$x^2 - 10x + 25 = 0$$

whence, by evolution,

$$x - 5 = \pm 0;$$

that is, $\quad x = 5 \pm 0 = 5 \text{ or } 5.$

1. Given $\dfrac{x}{x+1} + \dfrac{x+1}{x} = \dfrac{13}{6}$ to find the values of x.

OPERATION.

$$\frac{x}{x+1} + \frac{x+1}{x} = \frac{13}{6};$$

clearing of fractions, $\quad 6x^2 + 6x^2 + 12x + 6 = 13x^2 + 13x;$

reducing terms, $\quad x^2 + x = 6;$

adding $\left(\frac{1}{2}\right)^2$ to both members, $\quad x^2 + x + \frac{1}{4} = \frac{25}{4};$

by evolution, $\quad x + \frac{1}{2} = \pm \frac{5}{2};$

whence, $\quad x = 2 \text{ or } -3.$

Hence, for the solution of an affected quadratic equation we have the following

RULE.—I. *Reduce the given equation to the form of* $x^2 + 2ax = b$.

II. *Add to both sides of the equation the square of one-half the coefficient of* x, *and the first member will be a complete square.*

III. *Extract the square root of both members, and solve the resulting equation.*

290. When the equation has been reduced to the form of

$$x^2 + 2ax = b,$$

its roots may be obtained directly by the following obvious rule:

Write one-half the coefficient of x *with its contrary sign, plus or minus the square root of the second member increased by the square of one-half this coefficient.*

1. Given $x^2 - 6x = 55$ to find the values of x.

OPERATION.

$$x = 3 \pm \sqrt{55 + 9};$$

or, $$x = 3 \pm 8 = 11 \text{ or } -5, \textit{ Ans.}$$

EXAMPLES FOR PRACTICE.

Find the values of the unknown quantity in each of the following equations:

1. $x^2 + 2x = 15$. *Ans.* $x = 3$ or -5.
2. $x^2 - 6x = 16$. *Ans.* $x = 8$ or -2.
3. $x^2 - 20x = -96$. *Ans.* $x = 12$ or 8.
4. $x^2 - 6x - 7 = 33$. *Ans.* $x = 10$ or -4.
5. $x^2 - 28x + 80 = -115$. *Ans.* $x = 15$ or 13.
6. $x^2 + 6x + 1 = 92$. *Ans.* $x = 7$ or -13.
7. $x^2 + 12x = 589$. *Ans.* $x = 19$ or -31.
8. $x^2 - 6x + 10 = 65$. *Ans.* $x = 11$ or -5.
9. $x^2 + 12x + 2 = 110$. *Ans.* $x = 6$ or -18.
10. $x^2 - 14x = 51$. *Ans.* $x = 17$ or -3.
11. $x^2 + 20x + 19 = 0$. *Ans.* $x = -1$ or -19.
12. $x^2 - 6x + 6 = 9$. *Ans.* $x = 3 \pm 2\sqrt{3}$.
13. $x^2 + 8x = 12$. *Ans.* $x = -4 \pm 2\sqrt{7}$.
14. $x^2 + 12x = 10$. *Ans.* $x = -6 \pm \sqrt{46}$.
15. $3x^2 - 15x = -12$. *Ans.* $x = 4$ or 1.
16. $4x^2 + 12x = 40$. *Ans.* $x = 2$ or -5.
17. $2x^2 + 28 = 18x$. *Ans.* $x = 7$ or 2.

18. $(3x-5)(2x-2)=2(x^2+15)$. *Ans.* $x=5$ or -1.

19. $(2x+2)(5x-8)=(x+1)(5x+4)$. *Ans.* $x=4$ or -1.

20. $(3x+4)^2=54x$. *Ans.* $x=\frac{8}{3}$ or $\frac{2}{3}$.

21. $x^2-\frac{2x}{15}=\frac{7}{12}$. *Ans.* $x=\frac{5}{6}$ or $-\frac{7}{10}$.

22. $15x^2+\frac{2x}{3}=5$. *Ans.* $x=\frac{5}{9}$ or $-\frac{3}{5}$.

23. $4x^2-\frac{13x}{7}=\frac{5}{18}$. *Ans.* $x=\frac{7}{12}$ or $-\frac{5}{42}$.

24. $\frac{1}{2(x-1)}+\frac{3}{x^2-1}=\frac{1}{4}$. *Ans.* $x=5$ or -3.

25. $\frac{4}{x+1}+\frac{5}{x+2}=\frac{12}{x+3}$. *Ans.* $x=3$ or $-\frac{5}{3}$

SECOND METHOD OF COMPLETING THE SQUARE.

291. It frequently happens in reducing a quadratic to the form of $x^2+2ax=b$, that $2a$, the coefficient of x, becomes fractional, thus rendering the solution a little complicated. In such cases it will be sufficient to reduce the first member to the simplest entire terms. The equation will then be in the form

$$ax^2+bx=c \quad . \quad . \quad . \quad . \quad . \quad (1),$$

in which a and b are integral in form, and *prime to each other*, and c is entire or fractional.

To render the first term of (1) a perfect square, multiply both members by a; thus,

$$a^2x^2+abx=ac \quad . \quad . \quad . \quad . \quad . \quad (2).$$

Adding $\frac{b^2}{4}$ to both members,

$$a^2x^2+abx+\frac{b^2}{4}=ac+\frac{b^2}{4} \quad . \quad . \quad . \quad (3),$$

where the first member is a *complete square*. Now if b is even, $\frac{b^2}{4}$ will be entire; but if b is odd, $\frac{b^2}{4}$ will be fractional, a result which

we wish to avoid. To modify the rule to suit the latter case, suppose (3) to be multiplied by 4; thus,

$$4a^2x^2 \times 4abx + b^2 = 4ac + b^2 \quad . \quad . \quad . \quad (4).$$

The first member is now a complete square, and its terms are entire. Moreover, we observe that (4) may be obtained directly from (1) by multiplying (1) by $4a$, and adding b^2 to both sides of the result.

Hence, for the second method of completing the square in the first member, we have the following

RULE.—I. *Reduce the equation to the form of* $\mathrm{ax}^2 + \mathrm{bx} = \mathrm{c}$, *where* a *and* b *are prime to each other.*

II. *If* b *is even, multiply the equation by the coefficient of* x^2, *and add the square of one-half the coefficient of* x *to both members.*

III. *If* b *is odd, multiply the equation by 4 times the coefficient of* x^2, *and add the square of the coefficient of* x *to both members.*

The above rule may be considered as more general than the first; for if applied to equations in the form of $x^2 + 2ax = b$, the operation will be the same as by the first rule, with the simple modification of avoiding fractions in the first member, when $2a$ is fractional.

1. Given $5x^2-6x=8$, to find the values of x.

OPERATION.

	$5x^2 - 6x = 8.$
Multiplying by 5, and adding 3^2, or 9, to both members,	$25x^2 - 30x + 9 = 49;$
by evolution,	$5x - 3 = \pm 7;$
whence,	$5x = 10 \text{ or } -4;$
or,	$x = 2 \text{ or } -\frac{4}{5}.$

2. Given $15x^2-55x=350$, to find x.

OPERATION.

	$15x^2 - 55x = 350.$
Dividing by 5,	$3x^2 - 11x = 70;$
multiplying by 12, and adding 121 to both members,	$36x^2 - 132x + 121 = 961;$
by evolution,	$6x - 11 = \pm 31;$
whence,	$x = 7 \text{ or } -\frac{10}{3}.$

We will now apply this rule to an equation in the form of $x^2 + 2ax = b$, where $2a$ is *odd*.

3. Given $x^2 - 7x = 44$ to find x.

OPERATION.

	$x^2 - 7x = 44.$
Multiplying by 4 times 1, and adding 7^2 to both sides,	$4x^2 - 28x + 49 = 225;$
by evolution,	$2x - 7 = \pm 15;$
whence,	$x = 11$ or $-4.$

Thus we may always operate in such a manner as to avoid fractions in the first member; and indeed in the second member, if we first reduce both members of the equation to *entire terms*.

EXAMPLES FOR PRACTICE.

Solve each of the following equations :

1. $5x^2 + 4x = 204.$ *Ans.* $x = 6$ or $-\frac{34}{5}.$
2. $5x^2 + 4x = 273.$ *Ans.* $x = 7$ or $-\frac{39}{5}.$
3. $7x^2 - 20x = 32.$ *Ans.* $x = 4$ or $-\frac{8}{7}.$
4. $6x^2 + 15x = 9.$ *Ans.* $x = \frac{1}{2}$ or $-3.$
5. $2x^2 - 5x = 117.$ *Ans.* $x = 9$ or $-\frac{13}{2}.$
6. $21x^2 - 292x = -500.$ *Ans.* $x = 11\frac{19}{21}$ or $2.$
7. $6x^2 - 13x + 6 = 0.$ *Ans.* $x = \frac{2}{3}$ or $\frac{3}{2}.$
8. $7x^2 - 3x = 160.$ *Ans.* $x = 5$ or $-\frac{32}{7}.$
9. $3x^2 - 53x = -34.$ *Ans.* $x = 17$ or $\frac{2}{3}.$
10. $x^2 + 13x - 140 = 0.$ *Ans.* $x = 7$ or $-20.$
11. $3x^2 - 8x = 5 + 4\sqrt{3}.$ *Ans.* $x = 2 + \sqrt{3}$ or $\frac{2}{3} - \sqrt{3}.$
12. $x^2 + 11x - 80 = 0.$ *Ans.* $x = 5$ or $-16.$
13. $7x + \dfrac{72x}{10 - 3x} = 50.$ *Ans.* $x = 2$ or $\dfrac{250}{21}.$
14. $\dfrac{9 + 4x}{3} + \dfrac{x + 7}{x - 7} = x + 14.$ *Ans.* $x = 28$ or $9.$

15. $\frac{6(2x-11)}{x-3} = 26 - 4x.$ *Ans.* $x = 6$ or $\frac{1}{2}$.

16. $\frac{2x-3}{3x-5} + \frac{3x-5}{2x-3} = \frac{5}{2}.$ *Ans.* $x = \frac{7}{4}$ or 1.

17. $\frac{3x-2}{2x-5} + \frac{2x-5}{3x-2} = \frac{10}{3}.$ *Ans.* $x = \frac{13}{3}$ or $\frac{1}{7}$.

TREATMENT OF SPECIAL CASES.

292. Either of the two preceding rules is sufficient for the solution of any quadratic equation. There are certain cases, however, where the solution may be much simplified, either by a modification of one of the common rules, or by a special preparation of the equation.

293. When the coefficient of the highest power of the unknown quantity is a perfect square.

In this case the equation will be in the form of

$$a^2x^2 + bx = c \quad . \quad . \quad . \quad . \quad . \quad (1).$$

Let the quantity to be added to complete the square in the first member, be represented by t^2. Then

$$a^2x^2 + bx + t^2 = c + t^2 \quad . \quad . \quad . \quad . \quad (2).$$

Now in any binomial square, the middle term is twice the product of the square roots of the extremes. Hence,

$$2t \times ax = bx;$$

$$t = \frac{b}{2a};$$

$$t^2 = \left(\frac{b}{2a}\right)^2, \text{ or } \frac{b^2}{4a^2}.$$

And equation (2) becomes

$$a^2x^2 + bx + \left(\frac{b}{2a}\right)^2 = c + \left(\frac{b}{2a}\right)^2 \quad . \quad . \quad (3),$$

which may be used as the formula for completing the square, in this case. Or we may proceed according to the following

RULE.—*Divide the coefficient of* x *by twice the square root of the coefficient of* x^2, *and add the square of this result to both members.*

1. Given $25x^2 - 20x = -3$ to find the values of x.

In this example the number to be added to complete the square is

$$t^2 = \left(\frac{20}{2 \times 5}\right)^2 = (2)^2 = 4;$$

and the whole operation will be

$$25x^2 - 20x = -3,$$
$$25x^2 - 20x + 4 = 1,$$
$$5x - 2 = \pm 1,$$
$$x = \frac{3}{5} \text{ or } \frac{1}{5}, \textit{ Ans.}$$

2. Given $\frac{4x^2}{49} - \frac{x}{2} = \frac{51}{64}$ to find the values of x.

For this example we have

$$t^2 = \left(\frac{1}{2} \times \frac{7}{4}\right)^2 = \left(\frac{7}{8}\right)^2 = \frac{49}{64};$$

and the solution is as follows:

$$\frac{4x^2}{49} - \frac{x}{2} = \frac{51}{64},$$
$$\frac{4x^2}{49} - \frac{x}{2} + \frac{49}{64} = \frac{100}{64},$$
$$\frac{2x}{7} - \frac{7}{8} = \pm \frac{10}{8},$$
$$x = \frac{119}{16} \text{ or } -\frac{21}{16}, \textit{ Ans.}$$

EXAMPLES FOR PRACTICE.

1. $16x^2 + 12x = 10$. *Ans.* $x = \frac{1}{2}$ or $-\frac{5}{4}$.

2. $36x^2 - 5x = \frac{39}{144}$. *Ans.* $x = \frac{13}{72}$ or $-\frac{1}{24}$.

3. $81x^2 - 12x = -\frac{1}{3}$. *Ans.* $x = \frac{1}{9}$ or $\frac{1}{27}$.

4. $\frac{49x^2}{25} + \frac{6x}{5} = \frac{40}{49}$. *Ans.* $x = \frac{20}{49}$ or $-\frac{50}{49}$.

5. $\frac{841x^2}{625} - \frac{58x}{5} = 11$. *Ans.* $x = \frac{275}{29}$ or $-\frac{25}{29}$.

6. $\frac{7x^2 - 8x}{12} + \frac{7x^2 + 8x}{2} = 10x + 3$. *Ans.* $x = 2$ or $-\frac{18}{49}$.

294. When the equation is in the form of

$$x^2 + 2ax = (2a + m)m \quad . \quad . \quad . \quad (1).$$

In this case we may avoid tedious numerical operations, by the use of the auxiliary quantity, $2a$.

1. Given $x^2 - 5x = 6$ to find the values of x.

$$\text{Put} \quad 2a = 5; \quad \text{then} \quad 2a + 1 = 6;$$

and the equation becomes

$$x^2 - 2ax = 2a + 1;$$

whence,

$$x^2 - 2ax + a^2 = a^2 + 2a + 1,$$
$$x - a = \pm (a + 1),$$
$$x = 2a + 1 \text{ or } -1,$$
$$x = 6 \text{ or } -1, \textit{ Ans.}$$

2. Given $x^2 + 19x = 92$ to find x.

Assume $2a = 19$; then $4(2a + 4) = 8a + 16 = 92$;
and the solution will be as follows :

$$x^2 + 2ax = 8a + 16,$$
$$x^2 + 2ax + a^2 = a^2 + 8a + 16,$$
$$x + a = \pm (a + 4),$$
$$x = 4 \text{ or } -23, \textit{ Ans.}$$

Let it be observed that we always put the coefficient of x equal to $2a$. Then the method will apply, provided the second member is

$$2a + 1,$$
$$\text{or} \quad 4a + 4,$$
$$\text{or} \quad 6a + 9,$$
$$\text{or} \quad 8a + 16;$$

or, in general, $2am + m^2$; that is, *any multiple of* 2a *plus the square of the multiplier.*

EXAMPLES FOR PRACTICE.

Solve the following equations :

1. $x^2 - 7x = 8$. *Ans.* $x = 8$ or -1.
2. $x^2 + 11x = 26$. *Ans.* $x = 2$ or -13.
3. $x^2 - 17x = 60$. *Ans.* $x = 20$ or -3.
4. $x^2 + 21x = 46$. *Ans.* $x = 2$ or -23.

5. $x^2 - 75x = 76.$ *Ans.* $x = 76$ or $-1.$

6. $x^2 + 72x = 385.$ *Ans.* $x = 5$ or $-77.$

7. $x^2 - 325x = 3350.$ *Ans.* $x = 335$ or $-10.$

295. When large numerals may be avoided in the operation, by the use of an auxiliary quantity.

As all the examples of this kind cannot be included in any general classification, we give the following illustrations :

1. Given $x^2 + 9984x = 160000$ to find the values of x.

Put $2a = 10000$; then $2a - 16 = 9984$, and $32a = 160000$.

Whence,
$$x^2 + (2a - 16)x = 32a,$$
$$x^2 + (2a - 16)x + (a - 8)^2 = a^2 + 16a + 64,$$
$$x + (a - 8) = \pm (a + 8),$$
$$x = 16 \text{ or } -10000, \textit{ Ans.}$$

2. Given $x^2 + 45x = 9000$ to find x.

$$\text{Put} \quad a = 45; \quad \text{then} \quad 200a = 9000.$$
$$x^2 + ax = 200a,$$
$$4x^2 + 4ax + a^2 = a^2 + 800a,$$
$$2x + a = \pm\sqrt{a(a + 800)} = \sqrt{45 + 845}.$$

Multiply one of the factors under the radical by 5, and divide the other by 5 ; then we have
$$2x + a = \sqrt{225 \times 169},$$
$$2x + 15 \cdot 3 = \pm 15 \cdot 13,$$
$$2x = 15 \cdot 10 \text{ or } -15 \cdot 16,$$
$$x = 75 \text{ or } -120, \textit{ Ans.}$$

3. Given $16x^2 - 225x = 225$ to find x.

$$\text{Put} \quad 15 = a; \quad \text{then} \quad 16 = a + 1;$$

whence,
$$(a + 1)x^2 - a^2x = a^2,$$
$$4(a + 1)^2x^2 - 4a^2(a + 1)x + a^4 = a^4 + 4a^3 + 4a^2,$$
$$2(a + 1)x - a^2 = \pm (a^2 + 2a),$$
$$2(a + 1)x = 2a^2 + 2a \text{ or } -2a,$$
$$(a + 1)x = a(a + 1) \text{ or } -a,$$
$$x = 15 \text{ or } -\frac{15}{16}, \textit{ Ans.}$$

EQUATIONS IN THE QUADRATIC FORM.

296. There are many equations which, though not really quadratic, or of the second degree, may be solved by methods similar to those employed in quadratics. All such equations are reducible to the following form :

$$x^{2n} + 2ax^n = b;$$

in which x represents a simple or a compound quantity, and n is positive or negative, integral or fractional. It is always necessary *that the greater exponent should be twice the less.*

1. Given $x^4 - 16x^2 = -28$ to find the values of x.

OPERATION.

$$x^4 - 16x^2 = -28.$$

Adding 8^2, or 64, $x^4 - 16x^2 + 64 = 36$;
extracting the square root, $x^2 - 8 = \pm 6$;
by transposition, $x^2 = 14$ or 2;
whence, $x = \pm\sqrt{14}$ or $\pm\sqrt{2}$, *Ans.*

2. Given $x - 6x^{\frac{1}{2}} = -5$ to find the values of x.

OPERATION.

$$x - 6x^{\frac{1}{2}} = -5.$$

Completing the square, $x - 6x^{\frac{1}{2}} + 9 = 4$;
by evolution, $x^{\frac{1}{2}} - 3 = \pm 2$;
or, $x^{\frac{1}{2}} = 5$ or 1;
involving to the 2d power, $x = 25$ or 1, *Ans.*

3. Given $x^{-2} + 10x^{-1} = 24$ to find the values of x.

OPERATION.

$$x^{-2} + 10x^{-1} = 24,$$

Completing the square, $x^{-2} + 10x^{-1} + 25 = 49$;
extracting the square root, $x^{-1} + 5 = \pm 7$;
transposing, $x^{-1} = 2$ or -12;
taking the reciprocals, $x = \frac{1}{2}$ or $-\frac{1}{12}$, *Ans.*

4. Given $(x^2 + 2x)^2 - 23(x^2 + 2x) = -120$ to find the values of x.

This equation is in the quadratic form, for it contains the first and second powers of the compound quantity, $x^2 + 2x$.

For convenience, let us assume

$$y = x^2 + 2x.$$

Then by substitution in the given equation, we have

$$y^2 = 23y = -120;$$

whence,

$$y^2 - 23y + \tfrac{529}{4} = \tfrac{49}{4},$$

$$y - \tfrac{23}{2} = \pm \tfrac{7}{2},$$

$$y = 15 \text{ or } 8.$$

We have now the two equations,

$$x^2 + 2x = 15 \quad \text{and} \quad x^2 + 2x = 8,$$

which are solved as follows:

$$\begin{array}{rcl} x^2 + 2x = 15, & \qquad & x^2 + 2x = 8, \\ x^2 + 2x + 1 = 16, & & x^2 + 2x + 1 = 9, \\ x + 1 = \pm 4, & & x + 1 = \pm 3, \\ x = 3 \text{ or } -5. & & x = 2 \text{ or } -4. \end{array}$$

Thus the equation has four roots,

$$3, \quad -5, \quad 2, \quad -4;$$

and it will be found by trial that any one of these four values will satisfy the given equation.

Equations of the third and fourth degrees may often be solved like quadratics, even if they do not, at first, present themselves in the quadratic form, like the last equation. If any equation is susceptible of such a solution, it will be found to contain the *first and second powers* of some compound quantity, with known coefficients. To determine whether this be the fact in any particular case, we may proceed as follows:

Transpose all the terms to the first member; and if the highest power of the unknown quantity is not *even*, multiply the equation through by the unknown letter, to render it even. Then extract the square root to two or more terms, as the case may require; and if at any time a remainder occurs, which, with or without the absolute term, is a *multiple*, or an *aliquot part* of the root already obtained, a reduction to the quadratic form may be effected. Otherwise it will be impossible.

5. Given $x^4 - 4x^3 - 14x^2 + 36x + 45 = 0$ to find x.

SOLUTION.

$$\begin{array}{ll} & x^4 - 4x^3 - 14x^2 + 36x + 45 = 0 \;\underline{|\, x^2 - 2x} \\ & x^4 \\ 2x^2 - 2x & \overline{\quad - 4x^3 - 14x^2} \\ & \quad \underline{- 4x^3 + \;\; 4x^2} \\ & \qquad\qquad - 18x^2 + 36x + 45 \end{array}$$

Or by factoring, $\quad -18\,(x^2-2x) + 45$

Hence the given equation may be written thus:

$$(x^2 - 2x)^2 - 18\,(x^2 - 2x) + 45 = 0.$$

Without substituting any letter for the compound quantity, $x^2 - 2x$, the remaining part of the operation will be as follows:

$$(x^2 - 2x)^2 - 18\,(x^2 - 2x) = -45,$$
$$(x^2 - 2x)^2 - 18\,(x^2 - 2x) + 81 = 36,$$
$$(x^2 - 2x) - 9 = \pm 6,$$

whence, $\quad x^2 - 2x = 15 \quad . \;.\; . \quad (1),$

or, $\quad x^2 - 2x = \;\;3 \quad . \;.\; . \quad (2).$

From (1) we obtain $\quad x = 5$ or -3
" (2) " " $\quad x = 3$ or -1 } *Ans.*

6. Given $x^3 + 4ax^2 + 2a^2x - 4a^3 = 0$ to find x.

As the highest exponent of x is *odd*, we multiply the equation by x, and obtain

$$x^4 + 4ax^3 + 2a^2x^2 - 4a^3x = 0.$$

Taking the square root of two terms, and factoring the remainder,

$$(x^2 + 2ax)^2 - 2a^2\,(x^2 + 2ax) = 0.$$

Assume $y = x^2 + 2ax$; then

$$y^2 - 2a^2y = 0,$$
$$y^2 - 2a^2y + a^4 = a^4,$$
$$y - a^2 = \pm a^2,$$
$$y = 2a^2 \text{ or } 0.$$

Hence we have two equations, solved as follows:

$$x^2 + 2ax = 2a^2, \qquad\qquad x^2 + 2ax = 0,$$
$$x + a = \pm a\sqrt{3}, \qquad\qquad x^2 = -2ax,$$
$$x = -a \pm a\sqrt{3}, \qquad\qquad x = -2a.$$

Hence, $\quad x = -a\big(1 - \sqrt{3}\big), \quad -a\big(1 + \sqrt{3}\big),$ or $\; -2a$, *Ans.*

7. Given $25x^2 - 6 + \frac{1}{4x^2} = \frac{5}{4}$ to find x.

The two extremes in the first member are perfect squares. We will therefore seek for a *middle term* which will render the square complete. This will be *twice the products of the square roots of the extremes;* or

$$5x \times \frac{1}{2x} \times 2 = 5.$$

We therefore add 1 to both members, and solve as follows:

$$25x^2 - 5 + \frac{1}{4x^2} = \frac{9}{4},$$
$$5x - \frac{1}{2x} = \pm \frac{3}{2},$$
$$10x^2 - 1 = \pm 3x.$$
$$10x^2 \mp 3x = 1;$$
$$100x^2 \mp 30x + \tfrac{9}{4} = 10 + \tfrac{9}{4} = \tfrac{49}{4},$$
$$10x \mp \tfrac{3}{2} = \pm \tfrac{7}{2},$$
$$10x = 5 \text{ or } -2, \text{ or } 2 \text{ or } -5;$$
$$x = \pm \tfrac{1}{2} \text{ or } \mp \tfrac{1}{5}, \textit{ Ans.}$$

8. Given $x + 4\sqrt{x} = 21$ to find the values of x.

OPERATION.

$$x + 4\sqrt{x} = 21,$$
$$x + 4\sqrt{x} + 4 = 25,$$
$$\sqrt{x} + 2 = \pm 5,$$
$$\sqrt{x} = 3 \text{ or } -7,$$
$$x = 9 \text{ or } 49, \textit{ Ans.}$$

It should be observed here that when the equation contains a *radical,* as in the last example, it cannot be satisfied by the roots obtained, *without a trial of signs.* The roots found in the last solution are 9 and 49; but we have

$$\sqrt{9} = +3 \text{ or } -3,$$
$$\sqrt{49} = +7 \text{ or } -7.$$

Now we may verify the given equation, if we take $\sqrt{x} = +3$ or -7; but not otherwise.

Thus, putting $x=9$ and $\sqrt{x}=3$ in the given equation, we have

$$9+12=21;$$

also with $x=49$ and $\sqrt{x}=-7$, we have

$$49-28=21;$$

and the equation is satisfied in both cases.

But putting $x=9$ and $\sqrt{x}=-3$, we have

$$9-12=21;$$

also with $x=49$ and $\sqrt{x}=+7$, we have

$$49+28=21;$$

both of which are false.

In general, it will be found that a radical equation can be satisfied by each of the roots of solution, under at least *one* of the possible combinations of signs.

9. Given $2\sqrt{x}+\frac{2}{\sqrt{x}}=5$ to find x.

We have here a radical equation which is not in the quadratic form. In such cases, it is generally better to clear the equation of radical signs, either entirely or partially. Thus,

$$2\sqrt{x}+\frac{2}{\sqrt{x}}=5,$$

$$2x+2=5\sqrt{x},$$

$$2x-5\sqrt{x}=-2,$$

$$16x-40\sqrt{x}+25=9,$$

$$4\sqrt{x}-5=\pm 3,$$

$$4\sqrt{x}=8 \text{ or } 2,$$

$$\sqrt{x}=2 \text{ or } \frac{1}{2},$$

$$x=4 \text{ or } \frac{1}{4}, \textit{ Ans.}$$

EXAMPLES OF EQUATIONS SOLVED LIKE QUADRATICS.

1. $x^4-34x^2=-225$. *Ans.* $x=\pm 5$ or ± 3.
2. $x^6-35x^3+216=0$. *Ans.* $x=2$ or 3.
3. $x^6-4x^3-621=0$. *Ans.* $x=3$ or $\sqrt[3]{-23}$.
4. $x^{10}+31x^5-32=0$. *Ans.* $x=1$ or -2.

5. $x^3 - x^{\frac{3}{2}} = 56.$ *Ans.* $x = 4$ or $\sqrt[3]{49}.$

6. $x^{2n} - 2x^n = 8.$ *Ans.* $x = \sqrt[n]{4}$ or $\sqrt[n]{-2}.$

7. $20x^{\frac{2}{n}} - 31x^{\frac{1}{n}} = -12.$ *Ans.* $x = \left(\frac{3}{4}\right)^n$ or $\left(\frac{4}{5}\right)^n.$

8. $3\sqrt[3]{x^2} - 10\sqrt[3]{x} = -3.$ *Ans.* $x = 27$ or $\frac{1}{27}.$

9. $x + 5 - \sqrt{x+5} = 6.$ *Ans.* $x = 4$ or $-1.$

10. $(x+12)^{\frac{1}{2}} + (x+12)^{\frac{1}{4}} = 6.$ *Ans.* $x = 4$ or $69.$

11. $(x+a)^{\frac{1}{2}} + 2b(x+a)^{\frac{1}{4}} = 3b^2.$
Ans. $x = b^4 - a$ or $81b^4 - a.$

12. $x + \sqrt{5x+10} = 8.$ *Ans.* $x = 18$ or $3.$

13. $9x + 4 + 2\sqrt{9x+4} = 15.$ *Ans.* $x = \frac{5}{9}$ or $\frac{7}{3}.$

14. $\sqrt{10+x} - \sqrt[4]{10+x} = 2.$ *Ans.* $x = 6$ or $-9.$

15. $(x-5)^3 - 3(x-5)^{\frac{3}{2}} = 40.$ *Ans.* $x = 9$ or $5 + \sqrt[3]{(-5)^2}.$

16. $2(1 + x - x^2) - (1 + x - x^2)^{\frac{1}{2}} + \frac{1}{9} = 0.$

Ans. $x = \frac{1}{2} \pm \frac{1}{6}\sqrt{41}$ or $\frac{1}{2} \pm \frac{1}{3}\sqrt{11}.$

17. $x + 16 - 3\sqrt{x+16} = 10.$ *Ans.* $x = 9$ or $-12.$

18. $81x^2 + 17 + \frac{1}{x^2} = 99.$ *Ans.* $x = \pm 1$ or $\pm\frac{1}{9}.$

19. $25x^2 + 6 + \frac{4}{9x^2} = \frac{955}{9}.$ *Ans.* $x = \pm 2$ or $\pm\frac{1}{15}.$

20. $x^4 + 2x^3 - 7x^2 - 8x + 12 = 0.$
Ans. $x = 1, 2, -2,$ or $-3.$

21. $x^3 - 8x^2 + 19x - 12 = 0.$ *Ans.* $x = 1, 3,$ or $4.$

22. $x^4 - 10x^3 + 35x^2 - 50x + 24 = 0.$ *Ans.* $x = 1, 2, 3,$ or $4.$

23. $x^4 - 8ax^3 + 8a^2x^2 + 32a^3x - 9a^4 = 0.$
Ans. $x = a(2 \pm \sqrt{13})$ or $a(2 \pm \sqrt{3}).$

24. $y^4 - 2cy^3 + (c^2 - 2)y^2 + 2cy = c^2.$
Ans. $y = \frac{c}{2} \pm \left(\frac{c^2}{4} + 1 \pm \sqrt{1 + c^2}\right)^{\frac{1}{2}}.$

PROMISCUOUS EXAMPLES IN QUADRATICS.

1. $x^2 + 11x = 80.$ *Ans.* $x = 5$ or $-16.$

2. $3x - \dfrac{3x-3}{x-3} = \dfrac{3x-6}{2}.$ *Ans.* $x = 4$ or $-1.$

3. $\dfrac{x}{x+1} + \dfrac{x+1}{x} = \dfrac{13}{6}.$ *Ans.* $x = 2$ or $-3.$

4. $\dfrac{3}{2(x^2-1)} - \dfrac{1}{4(x+1)} = \dfrac{1}{8}.$ *Ans.* $x = 3$ or $-5.$

5. $\dfrac{x-1}{x+1} + \dfrac{x-2}{x+3} = \dfrac{2x+15}{x+19}.$ *Ans.* $x = 5$ or $-\dfrac{7}{4}.$

6. $\dfrac{x^3 - 10x^2 + 1}{x^2 - 6x + 9} = x - 3.$ *Ans.* $x = 1$ or $-28.$

7. $a^2 - 2ax + x^2 = b.$ *Ans.* $x = a \mp \sqrt{b}.$

8. $x^2 - 2ax + b^2 = 0.$ *Ans.* $x = a \pm \sqrt{a^2 - b^2}.$

9. $mx^2 - 2mx\sqrt{n} = nx^2 - mn.$ *Ans.* $x = \dfrac{\sqrt{mn}}{\sqrt{m} \mp \sqrt{n}}.$

10. $\dfrac{4x^2}{49} + \dfrac{8x}{21} = \dfrac{20}{3}.$ *Ans.* $x = 7$ or $-11\frac{2}{3}.$

11. $\dfrac{x^2}{361} - \dfrac{12x}{19} = -32.$ *Ans.* $x = 152$ or $76.$

12. $\dfrac{8}{(2x-4)^2} = 1 + \dfrac{16}{(2x-4)^4}.$ *Ans.* $x = 3$ or $1.$

13. $x^2 + 11 + \sqrt{x^2 + 11} = 42.$ *Ans.* $x = \pm 5$ or $\pm \sqrt{38}.$

14. $x^2 - 2x + 6\sqrt{x^2 - 2x + 5} = 11.$
Ans. $x = 1$ or $1 \pm 2\sqrt{15}.$

15. $x^4 + \dfrac{17x^3}{2} = 34x + 16.$ *Ans.* $x = 2, -2, -8,$ or $-\dfrac{1}{2}.$

16. $x - 1 = 2 + \dfrac{2}{\sqrt{x}}.$ *Ans.* $x = 4$ or $(-1)^2.$

17. $\dfrac{2\sqrt{x}+2}{4+\sqrt{x}} = \dfrac{4-\sqrt{x}}{\sqrt{x}}$ *Ans.* $x = 4$ or $7\frac{1}{9}.$

18. $x\sqrt{3\sqrt{2} - \dfrac{x^2}{2\sqrt{2}}} = \dfrac{2 + x^2}{\sqrt[4]{8}}$. *Ans.* $x = \pm(2 \pm \sqrt{2})^{\frac{1}{2}}$.

19. $\sqrt{x - \dfrac{1}{x}} - \sqrt{1 - \dfrac{1}{x}} = \dfrac{x-1}{x}$. *Ans.* $x = \dfrac{1}{2}(1 \pm \sqrt{5})$.

20. $\dfrac{1}{1 - \sqrt{1 - x^2}} - \dfrac{1}{1 + \sqrt{1 - x^2}} = \dfrac{\sqrt{3}}{x^2}$. *Ans.* $x = \pm \dfrac{1}{2}$.

21. $\left\{\dfrac{1}{1+x}\left(\dfrac{1}{1+x}\right)^{\frac{1}{2}}\right\}^{\frac{1}{3}} = \dfrac{\sqrt{2x}}{12}$. *Ans.* $x = 8$ or -9.

22. $\left(\dfrac{x + \sqrt{x^2 - 9}}{x - \sqrt{x^2 - 9}}\right)^{\frac{1}{2}} = x - 2$. *Ans.* $x = 5$ or 3.

23. $x^{\frac{6}{5}} + x^{\frac{3}{5}} = 756$. *Ans.* $x = 243$ or $-8\sqrt[3]{33614}$.

24. $6x^{2n} - 13x^{n} = -6$. *Ans.* $x = \sqrt[n]{\frac{3}{2}}$ or $\sqrt[n]{\frac{2}{3}}$.

25. $\sqrt{2 + 2x} + 2x = c(1 - x)$. *Ans.* $x = 1$ or $\dfrac{c^2 - 2}{(c + 2)^2}$.

26. $(x + a)^5 - (x - a)^5 = 352a^5$.
Ans. $x = \pm a\sqrt{5}$ or $\pm a\sqrt{-7}$.

27. $ax + \dfrac{1}{ax - 1} = \dfrac{b + c}{c}$. *Ans.* $x = \dfrac{1}{2ac}(2c + b \pm \sqrt{b^2 - 4c^2})$.

28. $\dfrac{1}{a + b + x} = \dfrac{1}{a} + \dfrac{1}{b} + \dfrac{1}{x}$. *Ans.* $x = -a$ or $-b$.

29. $\dfrac{2a}{x^2} + \dfrac{a^2 - x^2}{ax} = \dfrac{x^2 - a^2 + 16}{8a}$. *Ans.* $x = \pm a$.

30. $\dfrac{\sqrt{x + a} - \sqrt{x - a}}{\sqrt{x + a} + \sqrt{x - a}} = \dfrac{x}{2a}$. *Ans.* $x = \pm 2a\sqrt{\dfrac{1}{3}}$.

31. $\dfrac{a^2 + x^2}{a + x} + \dfrac{a^2 - x^2}{a - x} = 4a$. *Ans.* $x = \dfrac{a}{2}(1 \pm \sqrt{5})$.

32. $\dfrac{a + x + \sqrt{2ax + x^2}}{a + x} = b$. *Ans.* $x = \pm a\left\{\dfrac{1 \mp \sqrt{2b - b^2}}{\sqrt{2b - b^2}}\right\}$.

33. $\left(\dfrac{2 + x}{2 - x}\right)^2 = 1 + \dfrac{cx}{2b}$. *Ans.* $x = 2\left(1 \pm 2\sqrt{\dfrac{b}{c}}\right)$.

SIMULTANEOUS EQUATIONS CONTAINING QUADRATICS.

297. Having treated of quadratics containing only one unknown quantity, we will now consider certain cases of simultaneous equations where one or more are of a higher degree than the first.

298. In general, *the solution of two quadratic equations, involving two unknown quantities, depends upon the solution of a single equation of the fourth degree, containing one unknown quantity.*

To show this, let us represent the two equations under a general form, as follows:

$$x^2 + axy + by^2 + cx + dy + e = 0 \quad . \quad . \quad (1),$$
$$x^2 + a'xy + b'y^2 + c'x + d'y + e' = 0 \quad . \quad . \quad (2),$$

in which the coefficients a, b, c, etc., and a', b', c', etc., may have any value, positive or negative, integral or fractional.

Arranging the terms in these equations with reference to x, and factoring, we have

$$x^2 + (ay + c)x + by^2 + dy + e = 0 \quad . \quad . \quad (3),$$
$$x^2 + (a'y + c')x + b'y^2 + d'y + e' = 0 \quad . \quad . \quad (4).$$

Subtracting (4) from (3),

$$[(a - a')y + c - c']x + (b - b')y^2 + (d - d')y + (e - e') = 0.$$

By transposition,

$$[(a - a')y + c - c']x = (b' - b)y^2 + (d' - d)y + (e' - e);$$

whence,

$$x = \frac{(b' - b)y^2 + (d' - d)y + (e' - e)}{(a - a')y + (c - c')}.$$

This value of x substituted in (3) or (4), will give a final equation involving only y. Without actually making this substitution, which would lead to a complicated expression, it is obvious that the resulting equation would be of the fourth degree. For the value of x is in the form of

$$\frac{my^2 + ny + q}{ry + s},$$

in which y is involved to the second power. Therefore the term containing x^2 in (3), or (4), must involve y to the *fourth* power.

Hence, two equations, essentially quadratic, and containing two unknown quantities, cannot, in general, be solved by the rules for quadratics.

299. There are certain cases where simultaneous equations, involving one or more of the unknown quantities to a higher degree than the first, may be solved by means of a final equation in the quadratic form. Most of the examples of this kind are embraced in these three cases:

1st. Where one of the equations is simple, and the other quadratic.

2d. Where both of the equations are quadratic, but homogeneous.

3d. Where one or both of the equations are symmetrical, involving the different letters in a similar manner with respect to coefficients and exponents.

The following are illustrations:

1st. Simple and Quadratic Equations.

The solution is effected in this case by the ordinary methods of elimination.

1. Given $\left\{\begin{array}{l} 5x^2 - 6xy = 8 \\ 3x - 2y = 6 \end{array}\right\}$ to find x and y.

OPERATION.

$$5x^2 - 6xy = 8 \quad . \quad . \quad . \quad (1),$$
$$3x - 2y = 6 \quad . \quad . \quad . \quad (2).$$

From (2), $$x = \frac{6 + 2y}{3};$$

from (1), $$\frac{5(6 + 2y)^2}{9} - \frac{6y(6 + 2y)}{3} = 8,$$

$$180 + 120y + 20y^2 - 108y + 36y^2 = 72,$$
$$16y^2 - 12y = 108;$$
$$16y^2 - 12y + \frac{9}{4} = \frac{441}{4};$$
$$4y - \frac{3}{2} = \pm\frac{21}{2};$$
$$4y = 12 \text{ or } -9;$$

whence, $$y = 3 \text{ or } -\tfrac{9}{4},$$
and $$x = 4 \text{ or } \tfrac{1}{2}.$$

2d. Homogeneous Equations.

In the case of homogeneous equations, an auxiliary quantity is employed in the elimination.

2. Given $\left\{\begin{array}{l} 2x^2 - \quad xy = 6 \\ 2y^2 + 3xy = 8 \end{array}\right\}$ to find the values of x and y.

SOLUTION.

Put $x = vy$; then the given equations become

$$2v^2y^2 - vy^2 = 6, \quad \text{or} \quad y^2 = \frac{6}{2v^2 - v} \quad . \quad . \quad . \quad (1);$$

$$2y^2 + 3vy^2 = 8, \quad \text{or} \quad y^2 = \frac{8}{2 + 3v} \quad . \quad . \quad . \quad (2);$$

whence,

$$\frac{6}{2v^2 - v} = \frac{8}{2 + 3v},$$

$$6 + 9v = 8v^2 - 4v,$$

$$8v^2 - 13v = 6,$$

$$v = 2 \text{ or } -\tfrac{3}{8}.$$

Taking $v = 2$, equation (2) gives

$$y = \pm 1; \qquad \text{whence,} \qquad x = \pm 2.$$

Taking $v = -\frac{3}{8}$, the same equation gives

$$y = \pm \frac{8}{\sqrt{7}}; \qquad \text{whence,} \qquad x = \mp \frac{3}{\sqrt{7}}.$$

It may be observed here, that in this example, as in all other examples of simultaneous equations, the different *sets of values* which are capable of satisfying the equations, will be found by taking the signs *in their order;* that is, the upper signs should be taken throughout, and the lower signs throughout. Thus,

$$\begin{array}{lll} \text{when} & y = +1, & x = +2; \\ \text{"} & y = -1, & x = -2; \\ \text{"} & y = +\dfrac{8}{\sqrt{7}}, & x = -\dfrac{3}{\sqrt{7}}; \\ \text{"} & y = -\dfrac{8}{\sqrt{7}}, & x = +\dfrac{3}{\sqrt{7}}. \end{array}$$

3d. Symmetrical Equations.

When the equations are of this description, they may be solved by taking advantage of multiple forms, and of the necessary relations existing among the sum, difference, and product of two quantities.

3. Given $\begin{Bmatrix} x + y = 10 \\ xy = 21 \end{Bmatrix}$ to find the values of x and y.

OPERATION.

$$x + y = 10 \quad . \quad . \quad . \quad (1),$$
$$xy = 21 \quad . \quad . \quad . \quad (2).$$

Squaring (1), $\quad x^2 + 2xy + y^2 = 100;$
subtracting 4 times (2), $\quad x^2 - 2xy + y^2 = 16;$
by evolution, $\quad x - y = \pm 4;$
but in (1), $\quad x + y = 10;$
whence, $\quad x = 7$ or $3;$
$\quad y = 3$ or $7.$

4. Given $\begin{Bmatrix} x + \sqrt{xy} + y = 19 \\ x^2 + xy + y^2 = 133 \end{Bmatrix}$ to find the values of x and y.

OPERATION.

Put $\quad x + y = s, \quad$ and $\quad \sqrt{xy} = p.$

Then the given equations become

$$s + p = 19 \quad . \quad . \quad . \quad (1),$$
$$s^2 - p^2 = 133 \quad . \quad . \quad . \quad (2).$$

Dividing (2) by (1), $\quad s - p = 7 \quad . \quad . \quad . \quad (3);$
whence from (1) and (3), $\quad s = 13,$
and $\quad p = 6;$
that is, $\quad x + y = 13,$
and $\quad xy = 36.$

Proceeding now as in the third example, we have

$$x^2 + 2xy + y^2 = 169,$$
$$x^2 - 2xy + y^2 = 25,$$
$$x - y = \pm 5,$$
$$x = 9 \text{ or } 4.$$
$$y = 4 \text{ or } 9.$$

5. Given $\left\{\begin{matrix} x^{\frac{2}{3}} + y^{\frac{1}{5}} = 6 \\ x^{\frac{4}{3}} + y^{\frac{2}{5}} = 20 \end{matrix}\right\}$ to find x and y.

OPERATION.

Put $\quad x^{\frac{2}{3}} = P, \quad$ and $\quad y^{\frac{1}{5}} = Q;$

then $\quad x^{\frac{4}{3}} = P^2, \quad$ and $\quad y^{\frac{2}{5}} = Q^2.$

Substituting these values in the given equations,

$$P + Q = 6 \quad . \quad . \quad . \quad (1),$$

$$P^2 + Q^2 = 20 \quad . \quad . \quad . \quad (2).$$

From (1), $\quad P^2 + 2PQ + Q^2 = 36 \quad . \quad . \quad . \quad (3);$

taking (2) from (3), $\quad 2PQ = 16 \quad . \quad . \quad . \quad (4);$

taking (4) from (2), $\quad P^2 - 2PQ + Q^2 = 4;$

by evolution, $\quad P - Q = \pm 2;$

$$P + Q = 6;$$

$$P = 4 \text{ or } 2;$$

$$Q = 2 \text{ or } 4.$$

Hence,

$$x^{\frac{2}{3}} = 4 \text{ or } 2, \qquad y^{\frac{1}{5}} = 2 \text{ or } 4,$$

$$x^2 = 64 \text{ or } 8, \qquad y = 32 \text{ or } 1024.$$

$$x = \pm 8 \text{ or } \pm 2\sqrt{2};$$

In this example, the auxiliary letters were used to avoid fractional exponents in the operation. This practice, however, is not a necessity, but only a convenience. The auxiliary letters should be made to represent the *lowest powers* of the unknown quantities.

6. Given $\left\{\begin{matrix} x^2 + x^{\frac{4}{3}}y^{\frac{2}{3}} = 208 \\ y^2 + x^{\frac{2}{3}}y^{\frac{4}{3}} = 1053 \end{matrix}\right\}$ to find x and y.

OPERATION.

Assume $\quad x^{\frac{2}{3}} = P, \qquad y^{\frac{2}{3}} = Q;$

then $\quad x^{\frac{4}{3}} = P^2, \qquad y^{\frac{4}{3}} = Q^2;$

and $\quad x^2 = P^3, \qquad y^2 = Q^3.$

Substituting these values in the given equations, and factoring,

$$P^2(P + Q) = 208 = 13 \cdot 16 \quad . \quad . \quad . \quad (1),$$

$$Q^2(Q + P) = 1053 = 13 \cdot 81 \quad . \quad . \quad . \quad (2).$$

Dividing (2) by (1), $\frac{Q^2}{P^2} = \frac{81}{16}$ . . . . (3),

$$\frac{Q}{P} = \frac{9}{4} \quad . . . . (4);$$

or, $Q = \frac{9P}{4}$, and $P = \frac{4Q}{9}$ . . . . (5).

Substituting these values in (1) and (2), we have

$$P^3 + \frac{9P^3}{4} = 13 \cdot 16 \quad . . . (6),$$

$$Q^3 + \frac{4Q^3}{9} = 13 \cdot 81 \quad . . . (7).$$

From (6), $P^3 = x^2 = 64$;
from (7), $Q^3 = y^2 = 729$;
whence, $x = \pm 8$,
and $y = \pm 27$.

If we take the minus sign in the second member of equation (4), we shall obtain

$$x = \pm 8\sqrt{-\frac{13}{5}}, \qquad y = \pm 27\sqrt{-\frac{13}{5}}.$$

7. Given $\left\{\begin{matrix} x + y = 8 \\ x^3 + y^3 = 152 \end{matrix}\right\}$ to find the values of x and y.

OPERATION.

$$x + y = 8 \quad . . . (1),$$

$$x^3 + y^3 = 152 \quad . . . (2).$$

Cubing (1), $x^3 + 3x^2y + 3xy^2 + y^3 = 512$ . . . (3);
taking (2) from (3), $3x^2y + 3xy^2 = 360$ . . . (4),

$$xy(x + y) = 120 \quad . . . (5);$$

dividing (5) by (1), $xy = 15$ . . . (6).

Whence, by combining (1) and (6) as in the 3d example,

$$x = 5 \text{ or } 3, \qquad y = 3 \text{ or } 5.$$

300. For examples of more than two unknown quantities, no additional illustrations are necessary. The few cases which lead to a final equation in the quadratic form are to be treated by the same methods that apply to the preceding. And skill in the management of this whole class of examples, must depend less upon precept than upon practice.

301. As auxiliary to the solution of certain questions, particularly in geometrical progression, we give the following

PROBLEM.—Given $x+y=s$ and $xy=p$, to find the values of x^2+y^2, x^3+y^3, x^4+y^4, and x^5+y^5, expressed in terms of s and p.

SOLUTION.

$$x+y=s \;\ldots\ldots\ldots\ldots\ldots (1),$$

$$xy=p \;\ldots\ldots\ldots\ldots\ldots (2).$$

Squaring the first, $\quad x^2+2xy+y^2=s^2;$

$$2xy = 2p;$$

1st result, $\quad x^2+y^2=s^2-2p \;\ldots\ldots\ldots (A).$

Multiplying (A) by (1),

$$x^3+x^2y+xy^2+y^3=s^3-2ps;$$

subtracting $\quad xy(x+y) = ps;$

2d result, $\quad x^3+y^3=s^3-3ps \;\ldots\ldots\ldots (B).$

Again, squaring (A),

$$x^4+2x^2y^2+y^4=s^4-4s^2p+4p^2;$$

subtracting $\quad 2x^2y^2 = 2p^2;$

3d result, $\quad x^4+y^4=s^4-4s^2p+2p^2 \ldots (C).$

Multiplying (A) by (B),

$$x^5+x^3y^2+x^2y^3+y^5=s^5-5s^3p+6sp^2;$$

subtracting $\quad x^2y^2(x+y) = sp^2;$

4th result, $\quad x^5+y^5=s^5-5s^3p+5sp^2 \ldots (D).$

The following example will illustrate the use of these formulas.

1. Given $\left\{\begin{array}{l} x+y=9 \\ x^4+y^4=2417 \end{array}\right\}$ to find the values of x and y.

In this example we have

$$s=9, \quad s^2=81, \quad s^4=6561.$$

Hence, from (C), we have

$$6561-324p+2p^2=2417;$$

$$p^2-162p=-2072,$$

$$p^2-162p+6561=4489,$$

$$p-81=\pm 67,$$

$$xy=p=148 \text{ or } 14.$$

If we take $xy=148$, the values of x and y will be imaginary. Taking $xy=14$, with the equation $x+y=9$, we readily obtain

$$x=7 \text{ or } 2, \quad y=2 \text{ or } 7.$$

EXAMPLES OF SIMULTANEOUS EQUATIONS.

Find the values of the unknown quantities in the following groups of equations:

1. $\begin{cases} x - y = 15 \\ x - 2y^2 = 0 \end{cases}$. *Ans.* $\begin{cases} x = 18 \text{ or } 12\frac{1}{2}, \\ y = 3 \text{ or } -2\frac{1}{2}. \end{cases}$

2. $\begin{cases} xy + 2y^2 = 120 \\ 2x + y = 22 \end{cases}$. *Ans.* $\begin{cases} x = 8 \text{ or } 17\frac{2}{3}, \\ y = 6 \text{ or } -13\frac{1}{3}. \end{cases}$

3. $\begin{cases} x + y^2 = 25 \\ 4x = 9y \end{cases}$ *Ans.* $\begin{cases} x = 9 \text{ or } -14\frac{1}{16}, \\ y = 4 \text{ or } -6\frac{1}{4}. \end{cases}$

4. $\begin{cases} 5x^2 - y = 35 \\ 5x + y = 25 \end{cases}$. *Ans.* $\begin{cases} x = 3 \text{ or } -4, \\ y = 10 \text{ or } 45. \end{cases}$

5. $\begin{cases} 4x^2 + 3y^2 = 43 \\ 3x^2 - y^2 = 3 \end{cases}$. *Ans.* $\begin{cases} x = \pm 2, \\ y = \pm 3. \end{cases}$

6. $\begin{cases} 3x^2 + xy = 336 \\ 4x + y = 40 \end{cases}$. *Ans.* $\begin{cases} x = 28 \text{ or } 12, \\ y = -72 \text{ or } -8. \end{cases}$

7. $\begin{cases} xy + y^2 = 126 \\ 5(x + y) = 7x \end{cases}$. *Ans.* $\begin{cases} x = \pm 15, \\ y = \pm 6. \end{cases}$

8. $\begin{cases} x^2 + 4y^2 = 181 \\ 5(x - y) = 4y \end{cases}$. *Ans.* $\begin{cases} x = \pm 9, \\ y = \pm 5. \end{cases}$

9. $\begin{cases} x^2 + xy = 12 \\ y^2 + xy = 24 \end{cases}$. *Ans.* $\begin{cases} x = \pm 2, \\ y = \pm 4. \end{cases}$ ✓

10. $\begin{cases} x^2 - 2xy - y^2 = 1 \\ x + y = 2 \end{cases}$. *Ans.* $\begin{cases} x = \pm\sqrt{\frac{5}{2}}, \\ y = 2 \mp \sqrt{\frac{5}{2}}. \end{cases}$

11. $\begin{cases} x^2 + xy = 56 \\ xy + 2y^2 = 60 \end{cases}$. *Ans.* $\begin{cases} x = \pm 4\sqrt{2} \text{ or } \pm 14, \\ y = \pm 3\sqrt{2} \text{ or } \mp 10. \end{cases}$

12. $\begin{cases} 3x^2 + xy = 68 \\ 4y^2 + 3xy = 160 \end{cases}$. *Ans.* $\begin{cases} x = \pm 4 \text{ or } \pm \frac{34}{9}\sqrt{3}, \\ y = \pm 5 \text{ or } \mp \frac{16}{3}\sqrt{3}. \end{cases}$

13. $\begin{cases} x^2 + xy = 12 \\ xy - 2y^2 = 1 \end{cases}$. *Ans.* $\begin{cases} x = \pm 3 \text{ or } \pm \dfrac{8}{\sqrt{6}}, \\ y = \pm 1 \text{ or } \pm \dfrac{1}{\sqrt{6}}. \end{cases}$

14. $\begin{cases} x^2 - xy + y^2 = 21 \\ y^2 - 2xy + 15 = 0 \end{cases}$. *Ans.* $\begin{cases} x = \pm 4 \text{ or } \pm 3\sqrt{3}, \\ y = \pm 5 \text{ or } \pm \sqrt{3}. \end{cases}$

15. $\begin{cases} (x+y)^2 + 2(x+y) = 120 \\ xy - y^2 = 8 \end{cases}$.

Ans. $\begin{cases} x = 6,\ 9, \text{ or } -9 \pm \sqrt{5}; \\ y = 4,\ 1, \text{ or } -3 \mp \sqrt{5}. \end{cases}$

16. $\begin{cases} 6x^2 + 2y^2 = 5xy + 12 \\ 3y^2 - 3x^2 = 2xy + 3 \end{cases}$. *Ans.* $\begin{cases} x = \pm 2 \text{ or } \pm \dfrac{5}{\sqrt{31}}, \\ y = \pm 3 \text{ or } \mp \dfrac{6}{\sqrt{31}}. \end{cases}$

17. $\begin{cases} x^2 + y^2 = 65 \\ xy = 28 \end{cases}$. *Ans.* $\begin{cases} x = \pm 7 \text{ or } \pm 4, \\ y = \pm 4 \text{ or } \pm 7. \end{cases}$

18. $\begin{cases} x^2 + y^2 = 89 \\ x - y = 3 \end{cases}$. *Ans.* $\begin{cases} x = 8 \text{ or } -5, \\ y = 5 \text{ or } -8. \end{cases}$

19. $\begin{cases} x^3 + y^3 = 4914 \\ x + y = 18 \end{cases}$. *Ans.* $\begin{cases} x = 17 \text{ or } 1, \\ y = 1 \text{ or } 17. \end{cases}$

20. $\begin{cases} x^3 + y^3 = 189 \\ x^2y + xy^2 = 180 \end{cases}$. *Ans.* $\begin{cases} x = 5 \text{ or } 4, \\ y = 4 \text{ or } 5. \end{cases}$

21. $\begin{cases} (x^2 + y^2)(x - y) = 13 \\ xy(x - y) = 6 \end{cases}$. *Ans.* $\begin{cases} x = 3 \text{ or } -2, \\ y = 2 \text{ or } -3. \end{cases}$

22. $\begin{cases} x^3 + y^3 = (x+y)xy \\ x + y = 4 \end{cases}$. *Ans.* $\begin{cases} x = 2, \\ y = 2. \end{cases}$

23. $\begin{cases} 3x^2y^2 - 2xy = 1 \\ x = 2y \end{cases}$. *Ans.* $\begin{cases} x = \pm \sqrt{2} \text{ or } \pm\frac{1}{3}\sqrt{-6}, \\ y = \pm\frac{1}{2}\sqrt{2} \text{ or } \pm\frac{1}{6}\sqrt{-6}. \end{cases}$

24. $\begin{cases} x^3 + y^3 = 18xy \\ x + y = 12 \end{cases}$. *Ans.* $\begin{cases} x = 8 \text{ or } 4, \\ y = 4 \text{ or } 8. \end{cases}$

25. $\begin{cases} x^4 + y^4 = 2402 \\ x + y = 8 \end{cases}$. *Ans.* $\begin{cases} x = 7,\ 1, \text{ or } 4 \pm \sqrt{-105}, \\ y = 1,\ 7, \text{ or } 4 \mp \sqrt{-105}. \end{cases}$

26. $\begin{cases} x^2y + xy = 12 \\ x^3y + y = 18 \end{cases}$. *Ans.* $\begin{cases} x = 2 \text{ or } \frac{1}{2}, \\ y = 2 \text{ or } 16. \end{cases}$

27. $\begin{cases} x^3 + y^3 = 2xy(x+y) \\ xy = 16 \end{cases}$. *Ans.* $\begin{cases} x = \pm 2\sqrt{5} \pm 2, \\ y = \pm 2\sqrt{5} \mp 2. \end{cases}$

28. $\left\{\begin{array}{l} x^{\frac{3}{2}} + x^{\frac{3}{4}}y^{\frac{3}{4}} = a \\ y^{\frac{3}{2}} + x^{\frac{3}{4}}y^{\frac{3}{4}} = b \end{array}\right\}.$

Ans. $x = \left(\dfrac{a^4}{(a+b)^2}\right)^{\frac{1}{3}}, \quad y = \left(\dfrac{b^4}{(a+b)^2}\right)^{\frac{1}{3}}.$

29. $\left\{\begin{array}{r} x^2 - y^2 - (x+y) = 8 \\ (x+y)(x-y)^2 = 32 \end{array}\right\}.$ *Ans.* $\left\{\begin{array}{l} x = 5, \\ y = 3. \end{array}\right.$

30. $\left\{\begin{array}{l} x^2 + y^2 = a \\ xy \quad\;\; = b \end{array}\right\}.$ *Ans.* $\left\{\begin{array}{l} x = \pm \frac{1}{2}\sqrt{a+2b} \pm \frac{1}{2}\sqrt{a-2b}, \\ y = \pm \frac{1}{2}\sqrt{a+2b} \mp \frac{1}{2}\sqrt{a-2b}. \end{array}\right.$

31. $\left\{\begin{array}{l} x^2 - y^2 = 2a \\ xy \quad\;\; = 2b \end{array}\right\}.$ *Ans.* $\left\{\begin{array}{l} x = \left\{\;\; a \pm \sqrt{a^2+4b^2}\right\}^{\frac{1}{2}}, \\ y = \left\{- a \pm \sqrt{a^2+4b^2}\right\}^{\frac{1}{2}}. \end{array}\right.$

32. $\left\{\begin{array}{l} y\sqrt{x} + \sqrt{y} = 21 \\ xy^2 + y \quad\; = 333 \end{array}\right\}.$ *Ans.* $\left\{\begin{array}{l} x = 4 \text{ or } (\frac{1}{108})^2, \\ y = 9 \text{ or } 324. \end{array}\right.$

33. $\left\{\begin{array}{l} x + \sqrt{xy} = a \\ y + \sqrt{xy} = b \end{array}\right\}.$ *Ans.* $\left\{ x = \dfrac{a^2}{a+b}, \; y = \dfrac{b^2}{a+b}. \right.$

34. $\left\{\begin{array}{l} x \;\, + y \;\, = 35 \\ x^{\frac{1}{3}} + y^{\frac{1}{3}} = \;\, 5 \end{array}\right\}.$ *Ans.* $\left\{\begin{array}{l} x = 27 \text{ or } \;\, 8, \\ y = \;\, 8 \text{ or } 27. \end{array}\right.$

35. $\left\{\begin{array}{c} x + y = 10 \\ \sqrt{\dfrac{x}{y}} + \sqrt{\dfrac{y}{x}} = \dfrac{5}{2} \end{array}\right\}.$ *Ans.* $\left\{\begin{array}{l} x = 8 \text{ or } 2, \\ y = 2 \text{ or } 8. \end{array}\right.$

36. $\left\{\begin{array}{l} x^{\frac{1}{2}} + y^{\frac{1}{2}} = 4\left(x^{\frac{1}{2}} - y^{\frac{1}{2}}\right) \\ x \;\, - y \;\, = 16 \end{array}\right\}.$ *Ans.* $\left\{\begin{array}{l} x = 25, \\ y = \;\, 9. \end{array}\right.$

37. $\left\{\begin{array}{l} x^{\frac{3}{2}} + y^{\frac{2}{3}} = 3x \\ x^{\frac{1}{2}} + y^{\frac{1}{3}} = \;\, x \end{array}\right\}.$ *Ans.* $\left\{\begin{array}{l} x = 4 \text{ or } 1, \\ y = 8. \end{array}\right.$

38. $\left\{\begin{array}{r} x^{\frac{2}{3}} + y^{\frac{2}{3}} + 2x^{\frac{1}{3}} + 2y^{\frac{1}{3}} = 23 \\ x^{\frac{1}{3}}y^{\frac{1}{3}} = \;\, 6 \end{array}\right\}.$

Ans. $\left\{\begin{array}{l} x = 27, \;\; 8, \;\; -1, \text{ or } -216; \\ y = \;\, 8, 27, -216, \text{ or } \;\; -1. \end{array}\right.$

39. $\left\{\begin{array}{r} x^{\frac{4}{3}} + y^{\frac{2}{5}} + x^{\frac{2}{3}} + y^{\frac{1}{5}} = 26 \\ x^{\frac{2}{3}} y^{\frac{1}{5}} = 8 \end{array}\right\}.$

Ans. $\left\{\begin{array}{l} x = \pm 8, \ \pm 2\sqrt{2}, \text{ or } \left\{\frac{1}{2}\left(-7 \pm \sqrt{17}\right)\right\}^{\frac{3}{2}}; \\ y = 32, \quad 1024, \text{ or } \left\{\frac{1}{2}\left(-7 \mp \sqrt{17}\right)\right\}^{5}. \end{array}\right.$

40. $\left\{\begin{array}{l} \dfrac{x}{y} + \dfrac{4\sqrt{x}}{\sqrt{y}} = \dfrac{33}{4} \\ x - y \quad = 5 \end{array}\right\}.$ *Ans.* $\left\{\begin{array}{l} x = 9 \text{ or } \frac{605}{117}, \\ y = 4 \text{ or } \frac{20}{117}. \end{array}\right.$

41. $\left\{\begin{array}{l} x^{\frac{2}{3}} y^{\frac{3}{2}} \quad = 2y^2 \\ 8x^{\frac{1}{3}} - y^{\frac{1}{2}} = 14 \end{array}\right\}.$ *Ans.* $\left\{\begin{array}{l} x = 2744 \text{ or } 8, \\ y = 9604 \text{ or } 4. \end{array}\right.$

42. $\left\{\begin{array}{l} x^{\frac{3}{2}} + x^{\frac{3}{4}} y^{\frac{3}{4}} + y^{\frac{3}{2}} = 1009 \\ x^{3} + x^{\frac{3}{2}} y^{\frac{3}{2}} + y^{3} = 582193 \end{array}\right\}.$ *Ans.* $\left\{\begin{array}{l} x = 81 \text{ or } 16, \\ y = 16 \text{ or } 81. \end{array}\right.$

43. $\left\{\begin{array}{l} y^2 - 8x^{\frac{1}{2}} y \quad = 64 \\ y - 2x^{\frac{1}{2}} y^{\frac{1}{2}} = 4 \end{array}\right\}.$ *Ans.* $\left\{\begin{array}{l} x = 2\frac{1}{4}, \\ y = 16. \end{array}\right.$

44. $\left\{\begin{array}{l} x^2 y + x y^2 = 30 \\ \dfrac{1}{x} + \dfrac{1}{y} = \dfrac{5}{6} \end{array}\right\}.$ *Ans.* $\left\{\begin{array}{l} x = 3, \ 2, \quad 1, \text{ or } -6; \\ y = 2, \ 3, \ -6, \text{ or } \quad 1. \end{array}\right.$

45. $\left\{\begin{array}{l} x^2 + y^2 = 8 \\ \dfrac{1}{x^2} + \dfrac{1}{y^2} = \dfrac{1}{2} \end{array}\right\}.$ *Ans.* $\left\{\begin{array}{l} x = \pm 2, \\ y = \pm 2; \end{array}\right\}$ or $\left\{\begin{array}{l} x = \pm 2, \\ y = \mp 2. \end{array}\right.$

46. $\left\{\begin{array}{l} x^5 - y^5 = 3093 \\ x - y = 3 \end{array}\right\}.$ *Ans.* $\left\{\begin{array}{l} x = 5 \text{ or } -2, \\ y = 2 \text{ or } -5. \end{array}\right.$

47. $\left\{\begin{array}{l} x^2 + xy + y^2 = 7 \\ x^4 + x^2 y^2 + y^4 = 133 \end{array}\right\}.$ *Ans.* $\left\{\begin{array}{l} x = \pm 2 \text{ or } \mp 3, \\ y = \mp 3 \text{ or } \pm 2. \end{array}\right.$

48. $\left\{\begin{array}{l} \dfrac{1}{x} + \dfrac{1}{y} + \dfrac{1}{z} = 9 \\ \dfrac{2}{x} + \dfrac{3}{y} \qquad = 13 \\ 8x + 3y \qquad = 5 \end{array}\right\}.$ *Ans.* $\left\{\begin{array}{l} x = \dfrac{1}{2} \text{ or } \dfrac{5}{26}, \\ y = \dfrac{1}{3} \text{ or } \dfrac{15}{13}, \\ z = \dfrac{1}{4} \text{ or } \dfrac{15}{44}. \end{array}\right.$

THEORY OF QUADRATICS.

302. Having treated of the practical methods of solving quadratic equations, we will now proceed to consider certain general principles relating to quadratics.

303. Let us resume the general equation,

$$x^2 + 2ax = b \quad . \quad . \quad . \quad . \quad . \quad (A).$$

If we solve this equation, and represent one root by r and the other by r', we shall have

$$r = -a + \sqrt{a^2 + b} \quad . \quad . \quad . \quad (1),$$
$$r' = -a - \sqrt{a^2 + b} \quad . \quad . \quad . \quad (2).$$

By adding these equations, and also multiplying them together, we obtain

$$r + r' = -2a \quad . \quad . \quad . \quad . \quad . \quad (3),$$
$$rr' = -b \quad . \quad . \quad . \quad . \quad . \quad (4).$$

That is,

1. *The sum of the two roots is equal to the coefficient of* x *taken with the contrary sign.*

2. *The product of the two roots is equal to the absolute term taken with the contrary sign.*

304. From equations (3) and (4) in the last article, we have

$$2a = -(r + r'), \quad \text{and} \quad b = -rr'.$$

Substituting these values in (A), and transposing the absolute term, we have

$$x^2 - (r + r')x + rr' = 0;$$

or by factoring,

$$(x - r)(x - r') = 0. \qquad \text{Hence,}$$

If all the terms of a quadratic equation be transposed to the first member, the result will consist of two binomial factors, formed by annexing the two roots with their opposite signs to the unknown quantity.

305. A ***Quadratic Expression*** is one which contains the first and second powers of some letter or quantity.

By the principle established in the preceding article, any quadratic expression may be resolved into simple factors.

1. Let it be required to resolve the expression, $x^2 + 12x - 45$ into simple factors.

Assume $$x^2 + 12x - 45 = 0.$$

This equation readily gives

$$x = 3, \quad x = -15,$$

Hence, $$x^2 + 12x - 45 = (x - 3)(x + 15), \textit{ Ans.}$$

2. Separate $5x^2 - 8x + 3$ into simple factors.

We first separate the factor 5; thus,

$$5\left(x^2 - \frac{8x}{5} + \frac{3}{5}\right).$$

We may now factor the quantity within the parenthesis, as in the last example; thus,

$$x^2 - \frac{8x}{5} = -\frac{3}{5},$$

$$x^2 - \frac{8x}{5} + \frac{16}{25} = \frac{1}{25},$$

$$x - \frac{4}{5} = \pm\frac{1}{5},$$

$$x = 1 \text{ or } \frac{3}{5}.$$

And the given quantity is factored as follows:

$$5x^2 - 8x + 3 = 5(x - 1)(x - \tfrac{3}{5}), \textit{ Ans.}$$

EXAMPLES.

1. Resolve $x^2 + 2x - 120$ into simple factors.

Ans. $(x - 10)(x + 12)$.

2. Resolve $x^2 - 9x + 14$ into simple factors.

Ans. $(x - 2)(x - 7)$.

3. Resolve $x^2 + 8x + 15$ into simple factors.

Ans. $(x + 3)(x + 5)$.

4. Resolve $x^2 - 35x + 300$ into simple factors.

Ans. $(x - 15)(x - 20)$.

5. Resolve $x^2 - \frac{x}{4} - \frac{3}{8}$ into simple factors.

Ans. $\left(x - \frac{3}{4}\right)\left(x + \frac{1}{2}\right)$.

6. Resolve $15x^2 + 19x + 6$ into simple factors.

Ans. $15\left(x+\frac{2}{3}\right)\left(x+\frac{3}{5}\right)$.

7. Resolve $cx^2 - 2ax + c^3x - 2ac^2$ into simple factors.

Ans. $c\left(x-\dfrac{2a}{c}\right)(x+c^2)$.

306. The same principle also enables us to construct an equation whose roots shall be any given quantities. This is done by multiplying together the two binomial factors, which, according to the principle in question, the required equation must contain.

1. Find the equation whose roots shall be $\frac{1}{3}$ and $-\frac{1}{2}$.

Factors, $\begin{cases} x - \frac{1}{3} \\ x + \frac{1}{2}, \end{cases}$

Product, $x^2 + \dfrac{x}{6} - \dfrac{1}{6} = 0,$

or, $6x^2 + x - 1 = 0$, *Ans.*

EXAMPLES.

1. Find the equation whose roots shall be 6 and -15.

Ans. $x^2 + 9x - 90 = 0$.

2. Find the equation whose roots shall be 3 and -15.

Ans. $x^2 + 12x - 45 = 0$.

3. Find the equation whose roots shall be 16 and 9.

Ans. $x^2 - 25x + 144 = 0$.

4. Find the equation whose roots shall be 84 and -1.

Ans. $x^2 - 83x - 84 = 0$.

5. Find the equation whose roots shall be $\frac{2}{3}$ and $-\frac{1}{6}$.

Ans. $x^2 - \dfrac{x}{2} - \dfrac{1}{9} = 0$.

6. Find the equation whose roots shall be $\frac{7}{8}$ and $-\frac{4}{7}$.

Ans. $x^2 - \dfrac{17x}{56} - \dfrac{1}{2} = 0$.

7. Find the equation whose roots shall be $\frac{1}{2}$ and $\frac{1}{4}$.

Ans. $8x^2 - 6x + 1 = 0$.

8. Find the equation whose roots shall be $2a$ and $-c$.

Ans. $x^2 - (2a - c)\,x - 2ac = 0$.

DISCUSSION OF THE FOUR FORMS.

307. In the general equation $x^2 + 2ax = b$, the coefficient of x, as well as the absolute term, may be either positive or negative. Hence, to represent all the varieties, with respect to signs, we must employ the *four forms*, as follows:

$$x^2 + 2ax = +b \quad . \quad . \quad . \quad . \quad (1),$$
$$x^2 - 2ax = +b \quad . \quad . \quad . \quad . \quad (2),$$
$$x^2 + 2ax = -b \quad . \quad . \quad . \quad . \quad (3),$$
$$x^2 - 2ax = -b \quad . \quad . \quad . \quad . \quad (4).$$

From these equations we obtain

$$x = -a \pm \sqrt{a^2 + b} \quad . \quad . \quad . \quad (1),$$
$$x = +a \pm \sqrt{a^2 + b} \quad . \quad . \quad . \quad (2),$$
$$x = -a \pm \sqrt{a^2 - b} \quad . \quad . \quad . \quad (3),$$
$$x = +a \pm \sqrt{a^2 - b} \quad . \quad . \quad (4).$$

We may now consider what conditions will render these roots real or imaginary, positive or negative, equal or unequal.

308. Real and imaginary roots.

In the first and second forms, the quantity $a^2 + b$, under the radical, is positive, and the radical quantity is therefore real. But in the third and fourth forms, the quantity $a^2 - b$, under the radical, will be negative when b is numerically greater than a^2; in which case the radical quantity is imaginary. Hence,

1. *In each of the first and second forms, both roots are always real.*

2. *In each of the third and fourth forms, both roots are imaginary when the absolute term is numerically greater than the square of one-half the coefficient of* x; *otherwise they are real.*

309. Positive and negative roots.

Since $\quad a^2 + b > a^2$ and $a^2 - b < a^2$, we have

$$\sqrt{a^2 + b} > a \text{ and } \sqrt{a^2 - b} < a.$$

It follows, therefore, that the signs of the roots in the first and second forms will correspond to the signs of the radical; but the

signs of the roots in the third and fourth forms will correspond to the signs of the rational parts. Hence,

1. *In each of the first and second forms, one root is positive and the other negative.*

2. *In the third form both roots are negative, and in the fourth form both roots are positive.*

310. Equal and unequal roots.

It is obvious that in the first and second forms the two roots are always unequal; for in each of these forms, one root is numerically the *sum* of a rational and a radical part, and the other the *difference* of the same parts.

The same may be said of the third and fourth forms, if we except the case where $a^2 = b$; in which case the roots are equal, and we have, for the third form,

$$x = -a \pm 0 = -a \text{ or } -a,$$

and for the fourth form,

$$x = +a \pm 0 = +a \text{ or } +a.$$

Hence,

1. *In each of the first and second forms, the two roots are always unequal.*

2. *In each of the third and fourth forms, the roots will be equal when the absolute term is numerically equal to the square of one-half the coefficient of* x; *otherwise they will be unequal.*

In the first and third forms, the negative root consists of the sum of the rational and radical parts; while in the second and fourth forms, the positive root consists of the sum of the two parts. Hence, if we exclude the case of equal roots,

3. *In the first and third forms the negative root is numerically greater than the positive.*

4. *In the second and fourth forms, the positive root is numerically greater than the negative.*

The principles which we have now established, respecting the roots of quadratic equations, are all that are of importance, either theoretically or practically.

DISCUSSION OF PROBLEMS.

311. In the solution of particular problems involving quadratics, we shall find that in certain cases both roots of the equation will answer the conditions of the problem, while in other cases only one of the roots is admissible.

The reason is, that the algebraic expression is more general in its meaning than ordinary language; and thus the equation which represents the conditions of the given problem, will sometimes be found to represent the conditions of other analogous problems.

1. A man bought a horse for a certain price. Now if he sells him for $24, he will lose as much per cent. as the horse cost; required the price of the horse.

Let x denote the price. Then $x \times \frac{x}{100}$, or $\frac{x^2}{100}$, will be the loss, if he sells him for $24. Hence,

$$x - \frac{x^2}{100} = 24;$$

or $$x^2 - 100x = -2400;$$

$$x^2 - 100x + 2500 = 100;$$

whence, $$x - 50 = \pm 10;$$

or $$x = 60 \text{ or } 40.$$

Both values of x fulfill the conditions. For,

$$60 \times .60 = 36; \text{ and } 60 - 36 = 24.$$

$$40 \times .40 = 16; \text{ and } 40 - 16 = 24.$$

2. A person bought a number of sheep for $240; if he had bought 8 more for the same sum, each sheep would have cost $1 less. How many sheep did he purchase?

Let $x =$ the number of sheep purchased; then $\frac{240}{x} =$ cost of one. Had he purchased 8 sheep more, the cost of one would have been $\frac{240}{x+8}$. Hence, $$\frac{240}{x} - 1 = \frac{240}{x+8};$$

reducing, $$x^2 + 8x = 1920;$$

$$x^2 + 8x + 16 = 1936;$$

or, $$x + 4 = \pm 44;$$

whence, $$x = 40 \text{ or } -48.$$

In this case only the first value of x is admissible. The negative result, -48, is numerically the answer to the problem which would be formed by substituting in the above, the word *more* for the word *less*, and the word *less* for the word *more.*

INTERPRETATION OF IMAGINARY RESULTS.

312. We have seen that when the absolute term of a quadratic is negative, and numerically greater than the square of one-half the coefficient of the second power of the unknown quantity, the roots of the equation will be imaginary. Now the imaginary roots will always satisfy the equation, and it is necessary to ascertain what they indicate respecting the conditions of the problem which the equation represents.

1. Let it be required to divide 20 into two such parts, that their product shall be 140.

Let $x =$ one part; then $20 - x =$ the other. Hence,

$$x(20 - x) = 140;$$

or,

$$x^2 - 20x = -140,$$

$$x^2 - 20x + 100 = -40,$$

$$x - 10 = \pm\sqrt{-40},$$

$$x = 10 \pm \sqrt{-40}.$$

The result is *imaginary;* how shall it be interpreted? Recurring to the problem, we find that the greatest possible product that can be formed by multiplying together two parts of 20, is $10 \times 10 = 100$, the product of the halves of 20. Thus we find *that the problem is impossible.*

2. A farmer would enclose 50 square roods in a rectangular form, by a fence whose entire length shall be 24 rods. Required the length and breadth of the enclosure.

Let $x =$ the length, and $y =$ the breadth;

then

$$x + y = 12,$$

and

$$xy = 50;$$

$$x^2 + 2xy + y^2 = 144,$$

$$x^2 - 2xy + y^2 = -56,$$

$$x - y = \pm 2\sqrt{-14}.$$

whence,

$$x = 6 \pm \sqrt{-14}, \quad y = 6 \mp \sqrt{-14}.$$

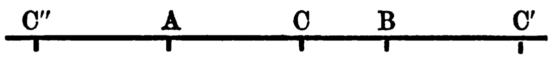
Thus, again, the results are imaginary. The problem, however, is impossible. For, if any given area is to be enclosed in a rectangular form, the perimeter will be the *least* when the figure is a *square*. But the square root of 50 exceeds 7; hence the field will have a perimeter of more than 28 rods, and cannot be enclosed by a fence 24 rods long. We conclude, therefore,

That imaginary roots indicate impossible conditions in the problem.

PROBLEM OF THE LIGHTS.

313. To illustrate more fully the rules of algebraic interpretation, we present for discussion the following general

PROBLEM.—Find upon the line which joins two lights, A and B, the point which is equally illuminated by them; admitting that the intensity of a light at any given distance, is equal to its intensity at the distance 1, divided by the square of the given distance.



Let a represent the intensity of the light A at the distance 1, and b the intensity of the light B at the distance 1.

Let c denote the distance AB, between the two lights. Assume A as the origin of distances, and regard all distances measured from A toward the right as positive.

Finally, let C denote the point of equal illumination, and let x represent the distance of this point from A. Then $c-x$ must be the distance of the same point from B. That is,

$$AC = x,$$
$$BC = c - x.$$

But by the conditions of the problem, the intensity of the light A at the distance x is $\frac{a}{x^2}$, and the intensity of the light B at the distance $c-x$ is $\frac{b}{(c-x)^2}$. But these intensities are equal, because

C represents the point of equal illumination; hence we have the equation,

$$\frac{a}{x^2} = \frac{b}{(c-x)^2} \quad . \quad . \quad . \quad . \quad . \quad (m);$$

or,
$$\frac{(c-x)^2}{x^2} = \frac{b}{a};$$

or,
$$\frac{c-x}{x} = \frac{\pm\sqrt{b}}{\sqrt{a}}.$$

By solving this equation, we obtain two values of x, as follows:

$$x = \frac{c\sqrt{a}}{\sqrt{a}+\sqrt{b}} \quad . \quad . \quad . \quad . \quad (1),$$

$$x = \frac{c\sqrt{a}}{\sqrt{a}-\sqrt{b}} \quad . \quad . \quad . \quad . \quad (2).$$

Since the two values of x are real, and also unequal, we conclude,

That there are two points of equal illumination on the line AB, *or on this line produced.*

This is evidently the conclusion to which we ought to arrive by an algebraic solution of the problem, in order to satisfy its conditions in a general manner. For, whatever may be the relative intensities of the two lights, there must always be one point of equal illumination *between* them. And if the lights are of unequal intensities, there must be another point of equal illumination, in the prolongation of the line, on the side of the lesser light.

We will now discuss the values of x, under several hypotheses.

1st. SUPPOSE $a > b$.

In this case, both values of x are positive; therefore, both points of equal illumination are situated to the right of A.

The first value of x is less than c; because $\frac{\sqrt{a}}{\sqrt{a}+\sqrt{b}}$ is less than unity, being a proper fraction. This value of x is also greater than *one half* of c; for we have

$$\sqrt{a} = \sqrt{a} \quad . \quad . \quad . \quad (1);$$

and since $a > b$, $\quad \sqrt{a}+\sqrt{b} < 2\sqrt{a} \quad . \quad . \quad . \quad (2).$

Dividing (1) by (2),

$$\frac{\sqrt{a}}{\sqrt{a}+\sqrt{b}} > \frac{1}{2};$$

therefore,

$$\frac{c\sqrt{a}}{\sqrt{a}+\sqrt{b}} > \frac{c}{2}.$$

Hence, the first point of equal illumination is at C, between A and B, but nearer B than A.

The second value of x is greater than c; for, $\frac{\sqrt{a}}{\sqrt{a}-\sqrt{b}}$ is greater than unity, being an improper fraction. Hence, the second point is at C′, in the prolongation of the line beyond B.

These conclusions are evidently correct. For, the supposition that a is greater than b, implies that B is the feebler light; both points should therefore be nearer B than A.

2d. SUPPOSE $a < b$.

The first value of x is positive. It is, moreover, *less than one half of c.* For

$$\sqrt{a} = \sqrt{a} \quad . \quad . \quad . \quad (1);$$

and since $a < b$,

$$\sqrt{a}+\sqrt{b} > 2\sqrt{a} \quad . \quad . \quad . \quad (2).$$

Dividing (1) by (2),

$$\frac{\sqrt{a}}{\sqrt{a}+\sqrt{b}} < \frac{1}{2};$$

therefore,

$$\frac{c\sqrt{a}}{\sqrt{a}+\sqrt{b}} < \frac{c}{2}.$$

Hence, the first point of equal illumination falls between A and B, and nearer A than B, as it should, because A is the lesser light.

The second value of x is negative, since the denominator, $\sqrt{a}-\sqrt{b}$, is negative.

Now in the statement of this problem, we considered distances reckoned from A toward the right as positive; hence, according to the rule for interpreting negative results, previously established (**181**), we must consider the negative result in this case, as a distance to be reckoned from A toward the left. Hence, the second point will be situated to the left of A, at C″. And this is as it should be, because A, under the present supposition, is the lesser light.

3d. SUPPOSE $a = b$.

In this case, the first value of x is positive, and equal to $\frac{c}{2}$. Hence the first point of equal illumination is midway between A and B.

The second value of x is $\frac{c\sqrt{a}}{0} = \infty$. This result indicates that there is no other point of equal illumination in the line AB, or in AB produced, at a finite distance from A.

These conclusions are obviously correct. For, under the present supposition, the two lights are equally intense. Hence any point, to be equally illuminated by them, must be equally distant from them; and the only point which fulfils this condition is the point midway between them.

If, however, we consider a and b as two varying quantities, at first unequal, but continually approaching equality, then the second value of x will become greater and greater by degrees, until it reaches infinity. Under these conditions, the second point of equal illumination will continually recede from A, moving toward the right or toward the left, according as a is greater or less than b, until it is finally removed to an infinite distance. In this view of the case, it is sometimes said that there are two points of equal illumination, under the hypothesis, $a = b$; one point being at an *infinite distance* from A.

4th. SUPPOSE $a = b$ AND $c = 0$.

The first value of x reduces to $\frac{0}{2\sqrt{a}} = 0$; hence the first point is situated at A.

The second value of x is $\frac{0}{0}$, the symbol of indetermination (**188**, 4). This result shows that there is an infinite number of other points equally illuminated by the two lights.

These interpretations are evidently correct. For, as the lights, under the present hypothesis, are equally intense, and both situated at A, every point in space must be equally illuminated by them.

5th. SUPPOSE $c = 0$, AND $a > b$ OR $a < b$.

Both values of x now reduce to 0; and the common rule for interpreting zero might lead us to suppose that the two points of

equal illumination coincide with the point A. But this conclusion is not strictly correct; for it is obvious that when two lights, of unequal intensities, occupy the same place, there is no point in space equally illuminated by them; not even the point in which they are both situated.

Let us return to the original equation (m), which truly represents the conditions of the problem. If we put $c=0$, the result is

$$\frac{a}{x^2}=\frac{b}{x^2};$$

an equation which cannot be satisfied *by any value of* x *whatever*, while $a > b$ or $a < b$. For by substituting any value for x we shall always obtain two unequal fractions. If $x = 0$, the two members are two *unequal infinities*.

We conclude, therefore, that under the supposition, $c = 0$, while a and b are unequal, the problem *fails* altogether, and is *impossible*.

Thus we learn that *zero* may be the answer to a possible, or an impossible problem. And whenever we obtain this symbol as the result of a solution, we must not interpret it on the assumption that the thing required in the problem is possible; but we must first determine whether the conditions are rational or absurd, by considering the nature of the problem, or by substituting zero in the original equation.

PROBLEMS PRODUCING QUADRATIC EQUATIONS.

314. It will be found that some of the following problems may be solved by a single unknown quantity, while others require two. Still others may be conveniently solved by means of either one or two letters. It is left to the judgment and skill of the learner to discover the mode of solution, in each example, which is most simple.

1. It is required to divide the number 14 into two such parts, that 9 times the quotient of the greater divided by the less, may be equal to 16 times the quotient of the less divided by the greater. *Ans.* 8 and 6.

2. A company, dining at an inn, agreed to pay \$3.50 for the entertainment; but before the bill was presented, two of the party left, in consequence of which each of the others had to pay 20 cents more than if all had been present. How many persons dined? *Ans.* 7.

3. Find a number, such that, if it be subtracted from 22, and the remainder multiplied by the number, the product will be 117. *Ans.* 13 or 9.

4. It is required to divide the number 18 into two such parts, that the squares of these parts may be to each other as 25 to 16. *Ans.* 10 and 8.

5. The difference of two numbers is 4, and their sum multiplied by the difference of their second powers gives 1600. What are the numbers? *Ans.* 12 and 8.

6. What two numbers are those whose difference is to the less as 4 to 3, and whose product multiplied by the less is equal to 504? *Ans.* 14 and 6.

7. A man purchased a field, whose length was to its breadth as 8 to 5. The number of dollars paid per acre was equal to the number of rods in the length of the field; and the number of dollars given for the whole was equal to 13 times the number of rods round the field. Required the length and breadth of the field. *Ans.* Length, 104 rods; breadth, 65 rods.

8. There is a stack of hay, whose length is to its breadth as 5 to 4, and whose height is to its breadth as 7 to 8. It is worth as many cents per cubic foot as there are feet in its breadth; and the whole is worth at that rate 224 times as many cents as there are square feet on the bottom. Required the dimensions of the stack.

Ans. Length, 20 feet; breadth, 16 feet; height, 14 feet.

9. There is a number, to which if you add 7 and extract the square root of the sum, and to which if you add 16 and extract the square root of the sum, the sum of the two roots will be 9. What is the number? *Ans.* 9.

NOTE.—Represent the number by $x^2 - 7$.

10. A and B together carried 100 eggs to market, and each received the same sum. If A had carried as many as B, he would

have received 18 pence for them; and if B had taken as many as A, he would have received 8 pence. How many had each?

Ans. A 40, and B 60.

11. The sum of two numbers is 6, and the sum of their cubes is 72. What are the numbers? *Ans.* 4 and 2.

12. A man traveled 36 miles in a certain number of hours; if he had traveled one mile more per hour, he would have required 3 hours less to perform his journey. How many miles did he travel per hour? *Ans.* 3 miles.

13. The sum of two numbers is 100, and the difference of their square roots is 2; what are the numbers? *Ans.* 36 and 64.

14. A gentleman bought a number of pieces of cloth for 675 dollars, which he sold again at 48 dollars a piece, and gained by the bargain as much as one piece cost him. What was the number of pieces? *Ans.* 15.

15. A merchant sold a piece of cloth for 39 dollars, and gained as much per cent. as it cost him. What did he pay for it?

Ans. \$30.

16. A merchant sent for a piece of goods and paid a certain sum for it, besides 4 per cent. for carriage; he sold it for \$390, and thus gained as much per cent. on the cost and carriage as the 12th part of the purchase money amounted to. What was the amount of the purchase money? *Ans.* \$300.

17. From two towns, 396 miles apart, two persons, A and B, set out at the same time, and traveled toward each other; after as many days as are equal to the difference of the number of miles they traveled per day, they met, when it appeared that A had traveled 216 miles. How many miles did each travel per day? *Ans.* A, 36; B, 30.

18. Divide the number 60 into two such parts that their product shall be 704. *Ans.* 44 and 16.

19. A vintner sells 7 dozen of sherry and 12 dozen of claret for £50, and finds that he has sold 3 dozen more of sherry for £10 than he has of claret for £6. Required the price of each.

Ans. Sherry, £2 per dozen; claret, £3.

20. A set out from C towards D, and traveled 7 miles a day. After he had gone 32 miles, B set out from D towards C, and went every day $\frac{1}{19}$ of the whole journey; and after he had traveled as many days as he went miles in a day, he met A. Required the distance from C to D. *Ans.* 76 or 152 miles.

21. A farmer received $24 for a certain quantity of wheat, and an equal sum at a price 25 cents less per bushel for a quantity of barley, which exceeded the quantity of wheat by 16 bushels. How many bushels were there of each?

Ans. 32 bushels of wheat and 48 of barley.

22. Two travelers, A and B, set out to meet each other, A leaving C at the same time that B left D. They traveled the direct road, and met 18 miles from the half-way point between C and D; and it appeared that A could have traveled B's distance in 15$\frac{3}{4}$ days, and B could have traveled A's distance in 28 days. Required the distance between C and D. *Ans.* 252 miles.

23. Find two numbers, whose difference, multiplied by the difference of their squares gives 32, and whose sum, multiplied by the sum of their squares gives 272. *Ans.* 5 and 3.

24. A and B hired a pasture at a certain rate per week, agreeing that each should pay according to the number of animals he should have in the pasture. At first A put in 4 horses, and B as many as cost him 18 shillings a week; afterward B put in 2 additional horses, and found that he must pay 20 shillings a week. At what rate was the pasture hired?

Ans. 30 shillings per week.

25. If a certain number be divided by the product of its two digits, the quotient will be 2; and if 27 be added to the number, the order of the digits will be inverted. What is the number?

Ans. 36.

26. It is required to find three numbers, such that the difference of the first and second shall exceed the difference of the second and third by 6, the sum of the numbers shall be 33, and the sum of the squares 441. *Ans.* 18, 9, and 6.

27. What two numbers are those whose product is 24, and whose sum added to the sum of their squares gives 62?

Ans. 4 and 6.

28. It is required to find two numbers, such that if their product be added to their sum, the result shall be 47; and if their sum be taken from the sum of their squares, the remainder shall be 62. *Ans.* 7 and 5.

NOTE.—In many examples of two unknown quantities, giving rise to *symmetrical* equations, it will be found convenient to denote one of the unknown quantities by $x + y$, and the other by $x - y$.

29. The sum of two numbers is 27, and the sum of their cubes is 5103. What are the numbers? *Ans.* 12 and 15.

30. The sum of two numbers is 9, and the sum of their fourth powers is 2417. What are the numbers? *Ans.* 7 and 2.

31. The product of two numbers multiplied by the sum of their squares, is 1248; and the difference of their squares is 20. What are the numbers? *Ans.* 6 and 4.

32. Two men are employed to do a piece of work, which they can finish in 12 days. In how many days could each do the work alone, provided it would take one 10 days longer than the other? *Ans.* One in 20 days; the other in 30 days.

33. The joint stock of two partners was \$1000; A's money was in trade 9 months, and B's 6 months; when they shared stock and gain, A received \$1140 and B \$640. What was each man's stock? *Ans.* A's, \$600; B's, \$400.

34. A speculator, going out to buy cattle, met with four droves. In the second were 4 more than 4 times the square root of one-half the number in the first; the third contained three times as many as the first and second; the fourth was one-half the number in the third, and 10 more; and the whole number in the four droves was 1121. How many were in each drove?
Ans. 1st, 162; 2d, 40; 3d, 606; 4th, 313.

35. Find two numbers, such that if the sum of their squares be subtracted from three times their product, 11 will remain; and if the difference of their squares be subtracted from twice their product, the remainder will be 14. *Ans.* 3 and 5.

36. Divide the number 20 into two such parts, that the product of their squares shall be 9216. *Ans.* 12 and 8.

37. Divide the number a into two such parts, that the product of their squares shall be b.

Ans. $\begin{cases} \text{Greater part, } & \frac{a}{2} + \frac{1}{2}\left(a^2 - 4\sqrt{b}\right)^{\frac{1}{2}}. \\ \text{Less part, } & \frac{a}{2} - \frac{1}{2}\left(a^2 - 4\sqrt{b}\right)^{\frac{1}{2}}. \end{cases}$

38. The greater of two numbers is a^2 times the less, and the product of the two is b^2. Find the numbers. *Ans.* $\frac{b}{a}$, and ab.

39. A certain number is equal to the product of three consecutive numbers; and if it be divided by each of them in turn, the sum of the quotients will be 74. What is the number?

Ans. $\begin{cases} 120; \text{ that is, } 4 \cdot 5 \cdot 6; \text{ or} \\ -120; \text{ that is, } (-4) \cdot (-5) \cdot (-6). \end{cases}$

40. An engraving whose length was twice its breadth was mounted on Bristol board, so as to have a margin 3 inches wide, and equal in area to the engraving, lacking 36 square inches. Find the width of the engraving. *Ans.* 12 inches.

41. A man has two square lots of unequal dimensions, containing together 25 A. 100 P. If the lots were contiguous to each other, it would require 280 rods of fence to embrace them in a single enclosure of six sides. Required the dimensions of the two lots. *Ans.* 62 rods and 16 rods, or 50 rods and 40 rods.

42. A person has £1300, which he divides into two portions, and lends at different rates of interest. He finds that the incomes from the two portions are equal; but if the first portion had been lent at the second rate of interest it would have produced £36, and if the second portion had been lent at the first rate of interest it would have produced £49. Find the rates of interest. *Ans.* 7 and 6 per cent.

43. A sets out from London to York, and B at the same time from York to London, both traveling uniformly. A reaches York 25 hours, and B reaches London 36 hours, after they have met on the road. Find in what time each has performed the journey. *Ans.* A, 55 hours; B, 66 hours.

44. A owns a village lot, in the form of a square, containing 36 square rods; B owns the adjacent lot on the same street, which is also a square, but greater than A's. Now if A should purchase all the front of B's lot, so as to extend the rear boundary line of his own through B's lot, parallel to the street, the two neighbors would possess equal quantities of land. Find the length of one side of B's lot. *Ans.* $6\left(1+\sqrt{2}\right)$ rods.

45. There are three numbers having the following relations to each other:—the sum of the squares of the first and second added to the first and second gives 32; the sum of the squares of the first and third added to the first and third gives 42; and the sum of the squares of the second and third added to the second and third gives 50. Required the qnantities.

$$\textit{Ans.} \begin{cases} \text{1st, } 3 \text{ or } -4; \\ \text{2d, } 4 \text{ or } -5; \\ \text{3d, } 5 \text{ or } -6. \end{cases}$$

46. What is the edge of that cube which contains as many solid units as there are linear units in the diagonal through its opposite corners. *Ans.* $\sqrt[4]{3}$.

47. It is required to find two quantities such that their sum, their product, and the sum of their squares, shall be equal to each other. *Ans.* $\frac{1}{2}\left(3 \pm \sqrt{-3}\right)$, and $\frac{1}{2}\left(3 \mp \sqrt{-3}\right)$.

48. Find two numbers whose sum, product, and the difference of whose squares, are equal to each other.

Ans. $\frac{1}{2}\left(3 \pm \sqrt{5}\right)$, and $\frac{1}{2}\left(1 \pm \sqrt{5}\right)$.

49. Find two numbers, such that their product shall be equal to the difference of their squares, and the sum of their squares shall be equal to the difference of their cubes.

Ans. $\pm \frac{1}{2}\sqrt{5}$, and $\frac{1}{4}\left(5 \pm \sqrt{5}\right)$.

SECTION VI.

PROPORTION, AND THE THEORY OF PERMUTATIONS AND COMBINATIONS.

PROPORTION.

315. Two quantities of the same kind may be compared, and their numerical relation determined, by finding how many times one contains the other. This mode of comparison gives rise to *ratio* and *proportion*.

316. The ***Ratio*** of two quantities is the quotient arising from dividing the first by the second.

There are two methods of indicating the ratio of two quantities.

1st. By writing the dividend before the divisor, with two dots between them; thus,

$$a : b$$

indicates the ratio of a to b, where a is the dividend and b the divisor.

2d. In the form of a fraction; thus, the ratio of a to b may be written

$$\frac{a}{b}.$$

317. A ***Compound Ratio*** is the product of two or more ratios. Thus,

Simple ratios, $\begin{cases} a : b \\ c : d \end{cases}$

Compound ratio, $ac : bd$.

318. The ***Duplicate Ratio*** of two quantities is the ratio of their squares.

319. The ***Triplicate Ratio*** of two quantities is the ratio of their cubes.

320. ***Proportion*** is an equality of ratios, both terms of each ratio being expressed. Thus, if two quantities, a and b, have the same ratio as two other quantities, c and d, the four

quantities, a, b, c, d, taken in their order, are said to be proportional.

Proportion may be written in two ways; thus,

$$a : b :: c : d,$$

which is read, *a is to b as c is to d*; or thus,

$$a : b = c : d,$$

which may be read as the other, or, *the ratio of a to b is equal to the ratio of c to d*. The second method of writing proportion is recommended as the more appropriate.

321. A ***Couplet*** consists of the two quantities which form a ratio.

322. The ***Terms*** of a proportion are the four quantities which are compared.

323. The ***Antecedents*** in a proportion are the first terms of the two couplets; or the *first* and *third* terms of the proportion.

324. The ***Consequents*** in a proportion are the second terms of the two couplets; or the *second* and *fourth* terms of the proportion.

325. The ***Extremes*** in a proportion are the *first* and *fourth* terms.

326. The ***Means*** in a proportion are the *second* and *third* terms.

327. When the first of a series of quantities has the same ratio to the second, as the second has to the third, as the third to the fourth, and so on, the several quantities are said to be in *continued proportion*, and any one of them is a *mean proportional* between the two adjacent ones. Thus, if

$$a : b = b : c = c : d = d : e;$$

then a, b, c, d, and e are in continued proportion, and b is a mean proportional between a and c, c a mean proportional between b and d; and so on.

328. One quantity is said to *vary* directly as another when the two quantities, by reason of their mutual dependence, have always a constant ratio, so that if one be changed the other will be changed in the same ratio.

Thus, for illustration, suppose, in the purchase of a commodity, a certain quantity, A, costs a certain sum, B. Now if the price of unity remain the same, it is evident that $2A$ will cost $2B$; $3A$ will cost $3B$; and in general, mA will cost mB. In this case the cost is said to vary directly as the quantity.

329. One quantity is said to vary *inversely* as another when the first has a constant ratio to the *reciprocal* of the other.

330. One quantity is said to vary as two others jointly, when it has a constant ratio to the product of the two.

331. The ***Sign of Variation*** is the symbol $\propto$; thus, the expression, $A \propto B$, signifies that A varies as B.

From the definition of variation, it is evident that the expression, $A \propto B$, is equivalent to the proportion,

$$A : B = m : 1,$$

where m is a *constant.* This proportion gives

$$A = mB.$$

Hence the general truth,

If A vary as B, then A is equal to B multiplied by some constant quantity.

PROPOSITIONS IN PROPORTION.

332. A ***Proposition*** is the statement of a truth to be demonstrated, or of a problem to be solved.

333. A ***Scholium*** is a remark showing the application or limitation of a preceding proposition.

334. If in the proportion

$$a : b = c : d,$$

the second method of indicating ratio be employed, we have

$$\frac{a}{b} = \frac{c}{d} \quad . \quad . \quad . \quad (A),$$

which is the *fundamental equation of proportion;* and any proposition relating to proportion will be *proved,* when shown to be consistent with this equation.

PROPOSITION I.—*In every proportion, the product of the extremes is equal to the product of the means.*

Let $a : b = c : d$, represent any proportion;

then by formula (A), $\frac{a}{b} = \frac{c}{d}$;

clearing of fractions, $bc = ad$.

That is, the product of b and c, the means, is equal to the product of a and d, the extremes.

SCHOLIUM.—From the last equation, we have

$$\left.\begin{array}{ll}\text{The first mean,} & b = \dfrac{ad}{c} \\ \text{The second mean,} & c = \dfrac{ad}{b}\end{array}\right\} \ldots (1).$$

$$\left.\begin{array}{ll}\text{The first extreme,} & a = \dfrac{bc}{d} \\ \text{The second extreme,} & d = \dfrac{bc}{a}\end{array}\right\} \ldots (2).$$

Hence,

1st. *Either mean is equal to the product of the extremes divided by the other mean* (1).

2d. *Either extreme is equal to the product of the means divided by the other extreme* (2).

PROPOSITION II.—Conversely:—*If the product of two quantities is equal to the product of two others, then either two may be taken for the means, and the other two for the extremes of a proportion.*

Let $ad = bc$.

Dividing by bd, $\frac{a}{b} = \frac{c}{d}$;

hence by formula (A), $a : b = c : d$,
in which the factors of the second product, bc, are the means, and the factors of the first product, ad, are the extremes.

PROPOSITION III.—*If four quantities are in proportion, they will be in proportion by* ALTERNATION; *that is, the antecedents will be to each other as the consequents.*

Let $a : b = c : d$;

then by formula (A), $\frac{a}{b} = \frac{c}{d} \ldots (1)$.

Multiplying (1) by b, $a = \frac{bc}{d}$ .... (2);

dividing (2) by c, $\frac{a}{c} = \frac{b}{d}$ ..... (3);

hence, $a : c = b : d$,

in which a and c, the antecedents of the given proportion, are proportional to b and d, the consequents of the given proportion.

PROPOSITION IV.—*If four quantities are in proportion, they will be in proportion by* INVERSION; *that is, the second will be to the first, as the fourth to the third.*

Let $a : b = c : d$;

then by formula (A), $\frac{a}{b} = \frac{c}{d}$;

clearing of fractions, $bc = ad$;

hence by Prop. II., $b : a = d : c$.

SCHOLIUM.—The last two propositions are but modifications of Prop. II. Thus we learn that from every equation three different forms of proportion may be derived.

Let $ad = bc$;
then $a : b = c : d$;
or, $a : c = b : d$;
or, $b : a = d : c$.

PROPOSITION V.—*Quantities which are proportional to the same quantities are proportioned to each other.*

If $a : b = m : n$ . . . (1),
and $c : d = m : n$ . . . (2),
we are to prove that $a : b = c : d$.

From (1), $\frac{a}{b} = \frac{m}{n}$;

from (2), $\frac{c}{d} = \frac{m}{n}$;

hence, $\frac{a}{b} = \frac{c}{d}$;

or $a : b = c : d$.

PROPOSITION VI.—*If four quantities are in proportion, they will be in proportion by* COMPOSITION *or* DIVISION; *that is, the sum or difference of the first and second will be to the second, as the sum or difference of the third and fourth is to the fourth.*

If $$a : b = c : d,$$
we are to prove that $$a : a \pm b = c : c \pm d.$$

By formula (A), $$\frac{a}{b} = \frac{c}{d} \quad \ldots\ldots \quad (1);$$

whence, $$1 + \frac{a}{b} = 1 + \frac{c}{d} \quad \ldots \quad (2),$$

or, $$\frac{a}{b} - 1 = \frac{c}{d} - 1 \quad \ldots \quad (3),$$

From (2), $$\frac{a+b}{b} = \frac{c+d}{d} \quad \ldots\ldots \quad (4);$$

from (3), $$\frac{a-b}{b} = \frac{c-d}{d} \quad \ldots\ldots \quad (5);$$

hence from (4), $$a + b : b = c + d : d;$$
and from (5), $$a - b : b = c - d : d.$$

SCHOLIUM.—In like manner it may be shown that
$$a + b : a = c + d : c,$$
$$a - b : a = c - d : c.$$

PROPOSITION VII.—*If four quantities are in proportion, the sum of the first and second is to their difference, as the sum of the third and fourth is to their difference.*

If $$a : b = c : d,$$
we are to prove that $$a + b : a - b = c + d : c - d.$$
By Prop. VI, $$a : a + b = c : c + d \quad \ldots \quad (1);$$
also, $$a : a - b = c : c - d \quad \ldots \quad (2).$$

from (1), $$\frac{a}{a+b} = \frac{c}{c+d} \quad \ldots\ldots \quad (3);$$

from (2), $$\frac{a}{a-b} = \frac{c}{c-d} \quad \ldots\ldots \quad (4);$$

dividing (4) by (3), $$\frac{a+b}{a-b} = \frac{c+d}{c-d} \quad \ldots\ldots \quad (5);$$

whence, $$a + b : a - b = c + d : c - d.$$

PROPOSITION VIII.—*If there be a proportion, consisting of three or more equal ratios, then either antecedent will be to its consequent, as the sum of all the antecedents is to the sum of all the consequents.*

Suppose $a : b = c : d = e : f = g : h =$, etc.

Then by comparing the ratio, $a : b$, first with itself, and afterward with each of the following ratios in succession, we obtain

$$ab = ba,$$
$$ad = bc,$$
$$af = be,$$
$$ah = bg, \text{ etc.};$$

whence, $a(b+d+f+h+\text{etc.}) = b(a+c+e+g+\text{etc.})$,

or, $a : b = a+c+e+g+\text{etc.} : b+d+f+h+\text{etc.}$

PROPOSITION IX.—*If four quantities are in proportion, the terms of either couplet may be multiplied or divided by any number, and the results will be proportional.*

Let $a : b = c : d$;

then, $\frac{a}{b} = \frac{c}{d}$.

And since the value of a fraction is not changed by multiplying or dividing both of its terms by the same number,

$$\frac{na}{nb} = \frac{c}{d} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1),$$

or, $$\frac{a}{b} = \frac{nc}{nd} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2),$$

in which n may be either integral or fractional. If n be integral, we have, from (1) and (2),

$$na : nb = c : d \quad . \quad . \quad . \quad . \quad (3),$$
$$a : b = nc : nd \quad . \quad . \quad . \quad . \quad (4),$$

in which the terms of the given couplets are multiplied. But put $n = \frac{1}{m}$; then (3) and (4) become

$$\frac{a}{m} : \frac{b}{m} = c : d \quad . \quad . \quad . \quad . \quad (5),$$

$$a : b = \frac{c}{m} : \frac{d}{m} \quad . \quad . \quad . \quad . \quad (6),$$

in which the given terms are divided.

PROPOSITION X.—*If four quantities are in proportion, either the antecedents or the consequents may be multiplied or divided by any number, and the results in every case will be proportional.*

Let $$a : b = c : d;$$

then $$\frac{a}{b} = \frac{c}{d} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1);$$

whence, $$\frac{a}{nb} = \frac{c}{nd} \quad . \quad . \quad . \quad . \quad . \quad . \quad (2);$$

or, $$\frac{na}{b} = \frac{nc}{d} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3);$$

in which n may be either integral or fractional. If n be integral, we have from (2) and (3),

$$a : nb = c : nd \quad . \quad . \quad . \quad . \quad (4),$$
$$na : b = nc : d \quad . \quad . \quad . \quad . \quad (5);$$

in which the given antecedents and consequents are multiplied. Put $n = \frac{1}{m}$; then (4) and (5) become

$$a : \frac{b}{m} = c : \frac{d}{m},$$

$$\frac{a}{m} : b = \frac{c}{m} : d;$$

in which the given antecedents and consequents are divided.

PROPOSITION XI.—*If four quantities which are in proportion, be multiplied or divided, term by term, by four other quantities also in proportion, the products, or quotients, taken in order, will be proportional.*

If $$a : b = c : d \quad . \quad . \quad . \quad (1),$$

and $$x : y = m : n \quad . \quad . \quad . \quad (2),$$

then we are to prove that

$$ax : by = cm : dn,$$

and $$\frac{a}{x} : \frac{b}{y} = \frac{c}{m} : \frac{d}{n}.$$

From (1) and (2), we obtain

$$ad = bc \quad . \quad . \quad . \quad . \quad (3),$$
$$xn = ym \quad . \quad . \quad . \quad . \quad (4);$$

multiplying (3) by (4), $(ax)(dn) = (by)(cm)$ . . (5);

dividing (3) by (4), $\left(\frac{a}{x}\right)\left(\frac{d}{n}\right) = \left(\frac{b}{y}\right)\left(\frac{c}{m}\right)$ . . (6);

whence, from (5), $ax : by = cm : dn$;

and from (6), $\frac{a}{x} : \frac{b}{y} = \frac{c}{m} : \frac{d}{n}$.

PROPOSITION XII.—*If four quantities are in proportion, like powers or roots of the same quantities will be in proportion.*

Let $a : b = c : d$;

then $\frac{a}{b} = \frac{c}{d}$ . . . . (1).

Raising (1) to the nth power, also taking the nth root of the same, $\frac{a^n}{b^n} = \frac{c^n}{d^n}$. . . . . . (2),

$$\frac{a^{\frac{1}{n}}}{b^{\frac{1}{n}}} = \frac{c^{\frac{1}{n}}}{d^{\frac{1}{n}}}. \quad . \; . \; . \; . \; . \; . \quad (3).$$

Hence from (2), $a^n : b^n = c^n : d^n$;

and from (3), $a^{\frac{1}{n}} : b^{\frac{1}{n}} = c^{\frac{1}{n}} : d^{\frac{1}{n}}$.

PROPOSITION XIII.—*If three quantities are in continued proportion, the product of the extremes is equal to the square of the mean.*

Let $a : b = b : c$;

then by Prop. I, $ac = bb = b^2$.

SCHOLIUM.—Taking the square root of the last equation, we have $b = \sqrt{ac}$; hence,

The mean proportional between two quantities is equal to the square root of their product.

PROPOSITION XIV.—*If three quantities are in continued proportion, the first is to the third, as the square of the first is to the square of the second; that is, in the duplicate ratio of the first and second.*

Let $a : b = b : c$;

then $b^2 = ac$;

multiplying by a, $ab^2 = a^2c$;

whence, by Prop. II, $a : c = a^2 : b^2$.

PROPOSITION XV.—*If four quantities are in continued proportion, the first is to the fourth, as the cube of the first is to the cube of the second; that is, in the triplicate ratio of the first and second.*

Let $a:b=b:c=c:d$;

then $ac=b^2$ . . . . . . (1),

and $c^2=bd$ . . . . . . (2);

multiplying (1) by (2), $ac^3=b^3d$;

whence, by Prop. II, $a:d=b^3:c^3$;

or, $a:d=a^3:b^3$.

PROBLEMS IN PROPORTION.

To show some of the applications of the preceding principles, we give the following problems:

1. Find two numbers, the greater of which shall be to the less as their sum to 42, and as their difference to 6.

Let $x=$ the greater, and $y=$ the less.

By the conditions, $\begin{cases} x:y=x+y:42 & \text{. . . . . (1),} \\ x:y=x-y:6 & \text{. . . . . (2).} \end{cases}$

Prop. V, $x+y:42=x-y:6$ . . . (3);

Prop. III, $x+y:x-y=42:6$ . . . (4);

Prop. VII, $2x:2y=48:36$ . . . (5);

Prop. IX, $x:y=4:3$ . . . (6).

From (1) and (6), Prop. V, $4:3=x+y:42$ . . (7);

" (2) " (6). " " $4:3=x-y:6$ . . . (8);

" (7), Prop. I, $x+y=56$;

" (8), " " $x-y=8$;

whence, $x=32$

and $y=24$, *Ans.*

2. Divide the number 14 into two such parts that the quotient of the greater divided by the less, shall be to the quotient of the less divided by the greater, as 16 to 9.

Let $x =$ the greater, then $14 - x =$ the less.

By the conditions, $$\frac{x}{14 - x} : \frac{14 - x}{x} = 16 : 9.$$

Multiplying terms, Prop. IX, $x^2 : (14 - x)^2 = 16 : 9$,
extracting square root, $x : 14 - x = 4 : 3$,
by composition, Prop. VI, $x : 14 = 4 : 7$,
dividing consequents, $x : 2 = 4 : 1$,

whence, $\left.\begin{array}{r} x = 8 \\ 14 - x = 6 \end{array}\right\}$, *Ans.*

3. There are three numbers in continued proportion; their sum is 52, and the sum of the extremes is to the mean as 10 to 3. Required the numbers.

Three numbers in continued proportion may be represented by x, xy, xy^2; for we observe that the product of the extremes will then be equal to the square of the mean. Hence,

by the conditions, $\left\{\begin{array}{l} x + xy + xy^2 = 52 \quad \ldots (1), \\ xy^2 + x : xy = 10 : 3 \quad \ldots (2). \end{array}\right.$

From (2), $y^2 + 1 : y = 10 : 3 \quad \ldots (3)$,
or, $y^2 + 1 : 2y = 10 : 6 \quad \ldots (4)$;
by Prop. VII, $y^2 + 2y + 1 : y^2 - 2y + 1 = 16 : 4$;
taking the square root, $y + 1 : y - 1 = 4 : 2$;
by Prop. VII, $2y : 2 = 6 : 2$;
or, $y : 1 = 3 : 1$;
whence, and from (1), $\left.\begin{array}{r} y = 3 \\ x = 4 \end{array}\right\}$, *Ans.*

4. The product of two numbers is 112; and the difference of their cubes is to the cube of their difference as 31 to 3. What are the numbers?

By the conditions, $\left\{\begin{array}{r} xy = 112 \quad \ldots (1), \\ x^3 - y^3 : (x - y)^3 = 31 : 3 \quad \ldots (2). \end{array}\right.$

From (2), Prop. IX, $x^2 + xy + y^2 : x^2 - 2xy + y^2 = 31 : 3 \quad \ldots (3)$;
by Prop. VI, $3xy : (x - y)^2 = 28 : 3 \quad \ldots (4)$;
by substitution, $336 : (x - y)^2 = 28 : 3 \quad \ldots (5)$;
whence, $(x - y)^2 = 36 \quad \ldots (6)$;
or, $x - y = 6 \quad \ldots (7)$.
From (1) and (7), we obtain $x = 14$, $y = 8$.

5. What two numbers are those whose difference is to their sum as 2 to 9, and whose sum is to their product as 18 to 77?

Let x and y represent the numbers.

By the conditions, $\begin{cases} x - y : x + y = 2 : 9 & \ldots (1), \\ x + y : \quad xy = 18 : 77 & \ldots (2). \end{cases}$

From (1), Prop. VII, $2x : 2y = 11 : 7 \quad \ldots (3),$

$$x = \frac{11y}{7} \quad \ldots (4).$$

From (2), by substitution, $\frac{18y}{7} : \frac{11y^2}{7} = 18 : 77 \quad \ldots (5);$

by Prop. IX, $18y : 11y^2 = 18 : 77 \quad \ldots (6);$

or, $18 \ : 11y = 18 : 77 \quad \ldots (7);$

or, $1 \ : \ y = 1 : 7 \quad \ldots (8);$

whence, $y = 7.$

6. Two numbers have such a relation to each other, that if 4 be added to each, the sums will be in the ratio of 3 to 4; and if 4 be subtracted from each, the remainders will be to each other as 1 to 4. What are the numbers? *Ans.* 5 and 8.

7. Divide the number 27 into two such parts, that their product shall be to the sum of their squares as 20 to 41.

Ans. 12 and 15.

8. In a mixture of rum and brandy, the difference between the quantities of the two is to the quantity of brandy, as 100 is to the number of gallons of rum; and the same difference is to the quantity of rum, as 4 to the number of gallons of brandy. How many gallons are there of each?

Ans. 25 of rum, and 5 of brandy.

9. There are two numbers whose product is 320; and the difference of their cubes is to the cube of their difference as 61 to 1. What are the numbers? *Ans.* 20 and 16.

NOTE.—In the last example, put $x + y =$ the greater, and $x - y =$ the less.

10. Divide 60 into two such parts, that their product shall be to the sum of their squares as 2 to 5. *Ans.* 40 and 20.

11. There are two numbers which are to each other as 3 to 2. If 6 be added to the greater and subtracted from the

less, the sum will be to the remainder as 3 to 1. What are the numbers? *Ans.* 24 and 16.

12. There are two numbers which are to each other as 16 to 9, and 24 is a mean proportional between them. What are the numbers? *Ans.* 32 and 18.

13. The sum of two numbers is to their difference as 4 to 1, and the sum of their squares is to the greater as 102 to 5. What are the numbers? *Ans.* 15 and 9.

14. The number 20 is divided into two parts, which are to each other in the duplicate ratio of 3 to 1. Find the mean proportional between these parts. *Ans.* 6.

15. There are two numbers in the ratio of 3 to 2; and if 6 be added to the greater, and subtracted from the less, the results will be as 9 to 4. What are the numbers? *Ans.* 39 and 26.

16. There are three numbers in continued proportion. The product of the first and second is to the product of the second and third, as the first is to twice the second; and the sum of the first and third is 300. What are the numbers?

Ans. 60, 120, and 240.

17. The sum of the cubes of two numbers is to the difference of their cubes, as 559 to 127; and the square of the first multiplied by the second gives 294. What are the numbers?

Ans. 7 and 6.

18. The cube of the first of two numbers is to the square of the second as 3 to 1, and the cube of the second is to the square of the first as 96 to 1. What are the numbers?

Ans. 12 and 24.

19. Given the proportion $(x+1)^4 : (x-1)^4 = 2(x+1)^2 : (x-1)^2$ to find the value of x.

Ans. $\dfrac{\sqrt{2}+1}{\sqrt{2}-1}$.

20. Prove that $a : b = c : d$, when

$$(a+b+c+d)\,(a-b-c+d) = (a-b+c-d)\,(a+b-c-d).$$

PERMUTATIONS AND COMBINATIONS.

335. The ***Permutations*** of things are the different results obtained by placing the things in every possible order. In forming permutations, all of the given elements, or a part only, may be taken at a time; but in any proposed system, the different results must contain the same number of things.

Thus, the permutations of the letters, a, b, c, taken two at a time, are

$$ab, ba, ac, ca, bc, cb.$$

The permutations of the same letters taken all at a time, are

$$cab, acb, abc, cba, bca, bac.$$

NOTE.—The results obtained by permuting things, where less than all are taken at a time, are sometimes called *variations* or *arrangements*; the word *permutations* would then be restricted to the case in which *all* the things are taken at a time.

336. The ***Combinations*** of things are the different collections that can be formed out of them, without regarding the order in which the things are placed, the same number of elements entering into all the results.

Thus, the combinations of the letters, a, b, c, taken two at a time, are

$$ab, ac, bc.$$

It will be observed that if the letters be regarded as *factors*, the combinations which may be formed by taking n at a time will constitute all the different products of the nth degree, of which the letters are capable.

337. To find the number of permutations of n things, taken r at a time.

Suppose the things to be n letters, $a, b, c, d \ldots.$

First: If we take each of the n letters by itself, there will be in every case $n-1$ other letters, or $n-1$ *reserved letters.*

Now if to each of the n letters we annex each of the reserved letters successively, we shall form all the permutations with *two*

letters each, of which the n letters are susceptible. But we shall obtain every time, $n-1$ results; thus,

with a,	ab,	ac,	$ad, \ldots,$	$(n-1$ results);
" b,	ba,	bc,	$bd, \ldots,$	" "
" c,	ca,	cb,	$cd, \ldots,$	" "
" d,	da,	db,	$dc, \ldots,$	" "
etc.		etc.		etc.

Now since there are n letters, each to be combined with $n-1$ reserved letters, there will be in all $n(n-1)$ results. That is,

The number of permutations of n *letters, taken two at a time, is* $n(n-1)$.

Second:—If we consider each of the permutations of the n letters with two in a set, apart from the other letters, there will be in every case $n-2$ reserved letters. Hence, to permute the n letters with *three* in a set, we shall have $n-2$ reserved letters, to be annexed successively to each of the $n(n-1)$ permutations with two in a set, thus forming $n(n-1)(n-2)$ new results. That is,

The number of permutations of n *letters, taken three at a time, is* $n(n-1)(n-2)$.

If the permutations of the n letters, taken $r-1$ at a time, were formed, there would be with respect to each, $n-(r-1)$, or $n-r+1$, reserved letters. And we might *conjecture* from the two preceding cases, that the number of permutations of n letters, taken r at a time, is

$$n(n-1)(n-2)\ldots(n-r+1) \quad . \quad . \quad . \quad (A),$$

or, *the product of the natural numbers from* n *down to* $n-r+1$, *inclusive.*

This may be demonstrated in a general manner, as follows:

Let x and x' represent any two consecutive numbers less than n, so that

$$x+1=x' \quad . \quad . \quad . \quad (1).$$

Let P represent the number of permutations of n letters, taken x in a set, and P' the number of permutations of the letters, taken $x+1$ or x' in a set.

Now if we consider each of the P permutations apart from the other letters, there will be in every case $n-x$ reserved letters. Thus we have $n-x$ reserved letters to be annexed successively to each of the P permutations, in order to form the P' permuta-

tions with $x+1$ or x' letters in a set. This will give $P(n-x)$ results; and we therefore have

$$P(n-x) = P' \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2).$$

Now we will show that if, according to the law already enunciated,

$$P = n(n-1)(n-2) \ldots . (n-x+1) \quad . \quad . \quad . \quad (3),$$

then the value of P' will be expressed by a similar formula. For, multiplying both members of (3) by $n-x$, and equating the result with the second member of (2),

$$P' = n(n-1)(n-2) \ldots . (n-x) \quad . \quad . \quad . \quad (4).$$

But from (1), $\qquad x = x' - 1.$

Substituting this value of x in (4),

$$P' = n(n-1)(n-2) \ldots . (n-x'+1) \quad . \quad . \quad . \quad (5).$$

Equations (3) and (5) are similar in form. Thus we have shown that if formula (A) holds when the letters are taken x at a time, it will hold when the letters are taken $x+1$ at a time. But it has been proved to hold when the letters are taken three at a time; hence it holds when they are taken four at a time; hence also it holds when they are taken five at a time, and so on. Thus it is true universally.

NOTE.—In the practical application of formula (A), it will be well to remember that the number of factors is equal to the number of letters taken in a set.

338. To find the number of permutations of n things taken all at a time.

Put $r = n$ in formula (A); the result will be

$$n(n-1)(n-2) \ldots . 1 \quad . \quad . \quad . \quad (B).$$

That is,

The number of permutations of n *things, taken all together in a set, is equal to the continued product of the natural numbers from* n *down to* 1, *inclusive.*

339. To find the number of combinations of n things, taken r at a time.

Let Z = the number of combinations of n things, taken r in a set;
P = the number of permutations of n things, taken r in a set;
P' = the number of permutations of r things, taken all together.

Now it is evident that all of the P permutations can be obtained, by subjecting the r things in each of the Z combinations to all the permutations of which they are susceptible. But a single combination of r things produces P' permutations, taking all the things in a set; hence the Z combinations will give $Z \times P'$ permutations, and we shall therefore have

$$Z \times P' = P;$$

whence,

$$Z = \frac{P}{P'}.$$

But by **337**, $P = n(n-1)(n-2)\ldots\ldots(n-r+1)$;

and by **338**, $P' = r(r-1)(r-2)\ldots\ldots 1.$

Hence,

$$Z = \frac{n(n-1)(n-2)\ldots\ldots(n-r+1)}{r(r-1)(r-2)\ldots\ldots 1} \quad \ldots \quad (C).$$

That is,

The number of combinations of n *letters, taken* r *at a time, is equal to the continued product of the natural numbers from* n *down to* n − r + 1 *inclusive, divided by the continued product of the natural numbers from* r *down to* 1 *inclusive.*

340. It is evident that for every combination of r things which we take out of n things, there will be left a combination of $n-r$ things. That is, every possible combination containing r things, corresponds to a combination of $n-r$ things which remain. Hence,

The number of combinations of n *things, taken* r *at a time, is equal to the number of combinations of* n *things, taken* n − r *at a time.*

This proposition may be demonstrated algebraically as follows:

Let Z represent the number of combinations of n things, taken r at a time, and Z' the number of combinations of n things, taken $n-r$ at a time. Let it be observed that the last factor in the numerator of Z' will be $n-(n-r)+1 = r+1$. Then

$$Z = \frac{n(n-1)(n-2)\ldots\ldots(n-r+1)}{r(r-1)(r-2)\ldots\ldots 1};$$

$$Z' = \frac{n(n-1)(n-2)\ldots\ldots(r+1)}{(n-r)(n-r-1)\ldots\ldots 1}.$$

By division we obtain

$$\frac{Z}{Z'}=\frac{n(n-1)(n-2)\ldots.1}{n(n-1)(n-2)\ldots.1}=1.$$

Whence, $$Z=Z'.$$

341. To find for what value of r, the number of combinations of n things taken r at a time is the greatest.

Consider r as a varying quantity, being at first unity, and changing to 2, 3, 4, ... successively.

Let Z represent the number of combinations for any value of r, and Z' the number of combinations for the succeeding value of r. We have

$$Z=\frac{n(n-1)(n-2)\ldots.(n-r+2)(n-r+1)}{r(r-1)(r-2)\ldots.1} \quad \ldots \quad (1).$$

And if in this equation we change r to $r+1$, the result must be the value of Z'; thus,

$$Z'=\frac{n(n-1)(n-2)\ldots.(n-r+1)(n-r)}{(r+1)(r)(r-1)\ldots.1} \quad \ldots \quad (2).$$

Dividing (2) by (1), observing that in the second member of (2), the factor which immediately precedes $(n-r+1)$ is $(n-r+2)$, we have

$$\frac{Z'}{Z}=\frac{n-r}{r+1};$$

whence, $$Z'=Z\times\left(\frac{n-r}{r+1}\right).$$

Now Z' is greater or less than Z, according as $\frac{n-r}{r+1}$ is greater or less than unity. That is, when

$$\frac{n-r}{r+1}>1,$$

the number of combinations will be increased by giving to r its succeeding value; but when

$$\frac{n-r}{r+1}<1,$$

the number of combinations will be diminished by giving to r its succeeding value.

But if $\qquad \frac{n-r}{r+1} > 1, \quad \text{then} \quad r < \frac{n-1}{2}.$

And if $\qquad \frac{n-r}{r+1} < 1, \quad \text{then} \quad r > \frac{n-1}{2}.$

Hence, that value of r which will give the greatest number of combinations, must not be less than $\frac{n-1}{2}$, or greater than $\frac{n-1}{2} + 1$, or $\frac{n+1}{2}$; hence, it will have one of the three values,

$$\frac{n-1}{2}, \quad \frac{n}{2}, \quad \frac{n+1}{2}.$$

1st. Suppose n *even*. Then the first and third values will be *fractional*, and therefore impossible for r; hence in this case

$$r = \frac{n}{2}.$$

2d. Suppose n *odd*. Then the second value will be fractional, and consequently impossible for r; hence, in this case r must have at least *one* of the other values. We will show that it may have either of them. For, suppose

$$r = \frac{n-1}{2}.$$

By **340**, the number of combinations will be the same, if

$$r = n - \frac{n-1}{2} = \frac{n+1}{2}.$$

That is, when n is odd, the greatest number of combinations will be obtained by making

$$r = \frac{n-1}{2} \quad \text{or} \quad r = \frac{n+1}{2},$$

the two values of r giving the same result.

EXAMPLES OF PERMUTATIONS AND COMBINATIONS.

1. How many different permutations may be formed of 10 letters, taken four at a time? *Ans.* 5040.

2. How many different permutations may be made of 6 things, taken all together in a set? *Ans.* 720.

3. How many different permutations may be made of 10 things, taken all together? *Ans.* 3628800.

4. How many different numbers can be formed with the five Arabic characters, 4, 3, 2, 1, 0; each of the characters occurring once, and only once in each number? *Ans.* 120.

5. How many different combinations may be formed of 8 things, taken 4 at a time? *Ans.* 70.

6. How many different combinations may be made of 16 things, taken 5 at a time? *Ans.* 4368.

7. How many different parties of 6 men each can be formed from a company of 20 men? *Ans.* 38760.

8. In how many different ways can a class of 6 boys be placed in line, one boy being denied the privilege of the head?

Ans. 600.

9. Find the greatest number of different products that can be formed with the prime numbers under 40, the products being all composed of the same number of factors. *Ans.* 1716.

10. The number of permutations of n things, taken 5 at a time, is equal to 120 times the number of combinations of the n things, taken 3 at a time; find n. *Ans.* $n = 8$.

11. At a certain house there were 8 regular boarders; and one of them agreed with the landlord to pay \$35 for his board so long as he could select from the company different parties, equal in number, to sit each for one day on a certain side of the table. At what price per day did he secure his board? *Ans.* \$.50.

12. A and B have each the same number of horses; and A can make up twice as many different teams by taking 3 horses together, as B can by taking 2 together. Required the number of horses that each has. *Ans.* 8.

13. There are 12 points in a plane, no three of which are in the same straight line with the exception of five, which are all in the same straight line. How many different straight lines can be formed by joining the points? *Ans.* 57.

SECTION VII.

SERIES.

342. A ***Series*** consists of a number of terms following one another, but so related that each may be derived from one or more of the preceding, by a fixed law.

A series may be finite or infinite, converging or diverging.

343. A ***Finite Series*** is one which by its law of development must terminate, or have only a finite number of terms.

344. An ***Infinite Series*** is one which by the law of its development can never terminate, but may have an infinite number of terms.

345. A ***Converging Series*** is an infinite series, the sum of whose terms is finite.

346. A ***Diverging Series*** is an infinite series, the sum of whose terms is infinite.

ARITHMETICAL PROGRESSION.

347. An ***Arithmetical Progression*** is a series of numbers or quantities increasing or decreasing from term to term by a common difference.

We may consider the common difference as a quantity continually *added*, in the algebraic sense; hence, it will be positive in an increasing series, and negative in a decreasing series. Thus,

$$1, \quad 3, \quad 5, \quad 7, \quad 9, \ldots$$

is an *increasing* arithmetical progression, in which the common difference is $+2$; and

$$20, \quad 18, \quad 16, \quad 14, \quad 12, \ldots$$

is a *decreasing* arithmetical progression, in which the common difference is -2.

348. To investigate the properties of an arithmetical progression, we may suppose the series to terminate; there will then be five parts or elements: the first term, the last term, the number of terms, the common difference, and the sum of the terms. The first term and last term are called the *extremes*, and all the terms between the extremes are called *arithmetical means.*

349. *In an Arithmetical Progression, the last term is equal to the first term plus the common difference multiplied by the number of terms less* 1.

Let a denote the first term, l the last term, d the common difference, and n the number of terms; then the series will be represented thus:

$$a, \quad (a+d), \quad (a+2d), \quad (a+3d), \ldots . \, l.$$

And we perceive that in every term the coefficient of d is equal to the number of *preceding* terms; hence,

$$l = a + (n-1)\,d \quad . \quad . \quad . \quad (A),$$

in which d is positive or negative, according as the series is an increasing or a decreasing one.

350. *In an arithmetical progression the sum of any two terms equidistant from the extremes is equal to the sum of the extremes.*

Let t denote a term of the series which has r terms before it, and t' a term which has r terms after it; then the terms, t and t', will be equidistant from the extremes. Suppose the series to be increasing; then from the nature of the series,

$$t = a + rd \quad . \quad . \quad . \quad (1),$$
$$t' = l - rd \quad . \quad . \quad . \quad (2);$$

whence, by addition,

$$t + t' = a + l.$$

351. *The sum of the terms of an Arithmetical Progression is equal to one half the sum of the two extremes, multiplied by the number of terms.*

Represent the sum of the series by S; then we have

$$S = a + (a+d) + (a+2d) + \ldots . + l \quad . \quad . \quad . \quad (1).$$

By writing the series in a reversed order, we have also

$$S = l + (l-d) + (l-2d) + \ldots . + a \quad . \quad . \quad . \quad (2).$$

Therefore, by addition,

$$2S = (a + l) + (a + l) + (a + l) + \dots + (a + l) \quad . \quad . \quad . \quad (3).$$

Now equation (3) expresses the sum of n terms, each equal to $(a + l)$; hence,

$$2S = n(a + l);$$

and dividing by 2, we obtain the formula,

$$S = \frac{n}{2}(a + l) \quad . \quad . \quad . \quad . \quad . \quad . \quad (B).$$

352. To insert any number of arithmetical means between two given terms.

Let n' denote the number of means to be inserted. Then the number of terms in the completed series will be $n' + 2$; and we shall have

$$n = n' + 2.$$

This value of n substituted in formula (A) (**349**), gives

$$l = a + (n' + 1)\,d;$$

whence,

$$d = \frac{l - a}{n' + 1} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (C).$$

Having the common difference, the means are readily obtained.

APPLICATION OF THE FORMULAS.

353. The two formulas,

$$l = a + (n - 1)\,d \quad . \quad . \quad . \quad . \quad (A),$$

$$S = \frac{n}{2}(a + l) \quad . \quad . \quad . \quad . \quad . \quad . \quad (B),$$

contain, in all, five quantities, a, l, n, d, S, four of which enter each equation. Now if any three of these quantities be given, the other two may be found; for, if the values of the three given quantities be substituted in the formulas, there will result two equations containing only two unknown quantities.

1. The first term of an arithmetical series is 5, the common difference 3, and the number of terms 24. Find the last term, and the sum of the series.

We have given, $a = 5, \quad d = 3, \quad n = 24;$

$$\left.\begin{array}{ll} \text{hence, by formula } (A), & l = 5 + (24-1)\,3 = 74 \\ \text{and by formula } (B), & S = \frac{24}{2}(5 + 74) \quad = 948 \end{array}\right\} \textit{Ans.}$$

2. Given $a = 15$, $d = -2$, and $S = 60$, to find the number of terms.

Substituting the given values in (A) and (B), we have

$$l = 15 - 2(n-1) \quad . \quad . \quad . \quad (1),$$

$$60 = \frac{n}{2}(15 + l) \quad . \quad . \quad . \quad . \quad . \quad (2);$$

whence, from (1), $\quad l = 17 - 2n,$

and from (2), $\quad l = \dfrac{120 - 15n}{n},$

$$\frac{120 - 15n}{n} = 17 - 2n,$$

$$120 - 15n = 17n - 2n^2,$$

$$n^2 - 16n = -60,$$

$$n - 8 = \pm 2,$$

$$n = 10 \text{ or } 6.$$

Both values of n are possible; for there are two series answering to the given conditions, one having 6 terms, and the other 10; these are

15, 13, 11, 9, 7, 5, 3, 1, -1, -3,

and 15, 13, 11, 9, 7, 5.

The sum of either series is 60.

EXAMPLES FOR PRACTICE.

1. The first term of an arithmetical series is 7, the common difference 3, and the number of terms 36; find the last term.
Ans. 112.

2. The first term of an arithmetical series is 275, the last term 5, and the number of terms 46; required the sum of the terms.
Ans. 6440.

3. The sum of an arithmetical series is 156, the number of terms 8, and the common difference 5. Required the extremes.

4. Find the sum of the terms in an arithmetical progression, knowing that the first term is 1, the common difference $\frac{1}{2}$, and the number of terms 101. *Ans.* 2626.

5. Find four arithmetical means between 7 and 37.
Ans. 13, 19, 25, 31.

6. The first term of an arithmetical series is 3, the number of terms 60, and the sum of the terms 3720; required the common difference, and the last term. *Ans.* $d = 2$, $l = 121$.

7. What will be the sum of the series if 9 arithmetical means be inserted between 9 and 109? *Ans.* 649.

8. If three arithmetical means be inserted between $\frac{1}{3}$ and $\frac{1}{2}$, what will be the common difference? *Ans.* $\frac{1}{24}$.

9. What debt can be discharged in a year by paying 1 cent the first day, 3 cents the second, 5 cents the third, and so on, increasing the payment each day by 2 cents? *Ans.* \$1332.25.

10. A footman travels the first day 20 miles, 23 the second, 26 the third, and so on, increasing the distance each day 3 miles. How many days must he travel at this rate to go 438 miles?
Ans. 12.

11. Find the sum of n terms of the progression 1, 2, 3, 4, 5, 6, ...
Ans. $S = \frac{n}{2}(1 + n)$.

12. Find the sum of n terms of the progression 1, 3, 5, 7, ...
$S = n^2$.

13. The sum of the terms of an arithmetical series is 950, the common difference is 3, and the number of terms 25. What is the first term? *Ans.* 2.

14. A man bought a certain number of acres of land, paying for the first \$$\frac{1}{3}$; for the second, \$$\frac{2}{3}$; and so on. When he came to settle he had to pay \$3775. How many acres did he purchase?
Ans. 150 acres.

15. The 14th, 134th, and last terms of an arithmetical progression are 66, 666, and 6666, respectively. Required the number of terms. *Ans.* 1334.

THE TEN CASES.

354. Given any three of the quantities, *a, l, n, d, S,* to find the other two.

This problem will present *ten cases*, each giving rise to two formulas, making in all twenty different formulas, or four values for each letter. The results in each case may be obtained directly from the two fundamental equations, or those of any particular case may be derived from some preceding case, as is most convenient. The whole will be left as an exercise for the student.

No.	Given.	To find.	Formulas.	
1	a, d, n	$l,\ S$	$l=a+(n-1)\,d;$	$S=\frac{1}{2}n[2a+(n-1)d].$
2	l, d, n	$a,\ S$	$a=l-(n-1)\,d;$	$S=\frac{1}{2}n\,[2l-(n-1)d].$
3	a, n, l	$d,\ S$	$d=\dfrac{l-a}{n-1};$	$S=\frac{1}{2}n\,(a+l).$
4	d, n, S	$a,\ l$	$a=\dfrac{2S-n\,(n-1)\,d}{2n};$	$l=\dfrac{2S+n\,(n-1)\,d}{2n}.$
5	a, n, S	$d,\ l$	$d=\dfrac{2\,(S-an)}{n\,(n-1)};$	$l=\dfrac{2S}{n}-a.$
6	l, n, S	$d,\ a$	$d=\dfrac{2\,(nl-S)}{n\,(n-1)};$	$a=\dfrac{2S}{n}-l.$
7	a, d, l	$n,\ S$	$n=\dfrac{l-a}{d}+1;$	$S=\dfrac{(l+a)\,(l-a+d)}{2d}.$
8	a, l, S	$n,\ d$	$n=\dfrac{2S}{a+l};$	$d=\dfrac{(l+a)\,(l-a)}{2S-(l+a)}.$
9	a, d, S	$n,\ l$	$n=\dfrac{d-2a\pm\sqrt{(d-2a)^2+8dS}}{2d};$	$l=a+(n-1)d.$
10	l, d, S	$n,\ a$	$n=\dfrac{d+2l\pm\sqrt{(d+2l)^2-8dS}}{2d};$	$a=l-(n-1)d.$

PROBLEMS IN ARITHMETICAL PROGRESSION

TO WHICH THE FORMULAS DO NOT IMMEDIATELY APPLY.

355. When in the conditions of a problem no three of the five parts, a, l, n, d, S, are directly given, the general formulas will not directly apply. It is usually necessary in such instances to represent the several terms of the series by means of two or more unknown quantities; and for this purpose there are two methods of notation.

1st. Let x denote the first term and y the common difference; thus,

$$x, (x + y), (x + 2y), (x + 3y) \ldots .$$

This method of notation, however, is seldom the most expedient.

2d. When the number of terms is odd, denote the *middle* term by x, and the common difference by y; then we shall have,

for three terms, $(x - y), x, (x + y)$;
for five terms, $(x - 2y), (x - y), x, (x + y), (x + 2y)$.

And when the number of terms is even, represent the two middle terms by $x - y$ and $x + y$ respectively, $2y$ being the common difference; thus,

$$(x - 3y), (x - y), (x + y, (x + 3y).$$

The advantage of the second method is, that the sum of all the terms, or the sum and difference of two terms equidistant from the extremes, will each contain but a single unknown quantity.

1. There are three numbers in arithmetical progression; the sum of these numbers is 18, and the sum of their squares is 158. What are the numbers? *Ans.* 1, 6, 11.

2. There are five numbers in arithmetical progression; their sum is 65, and the sum of their squares 1005. What are the numbers? *Ans.* 5, 9, 13, 17, 21.

3. It is required to find four numbers in arithmetical progression, such that their common difference shall be 4, and their continued product 176985. *Ans.* 15, 19, 23, 27.

4. There are four numbers in arithmetical progression; the sum of the extremes is 8, and the product of the means 15. What are the numbers? *Ans.* 1, 3, 5, 7.

5. A person starts from a certain place and goes 1 mile the first day, 2 the second, 3 the third, and so on; in six days after, another sets out from the same place in pursuit, and travels uniformly 15 miles a day. How many days after the second starts before they are together? *Ans.* 3 days, and 14 days.

NOTE.—Reconcile these two values.

6. A man has borrowed $60. What sum shall he pay daily to cancel the debt in 60 days; interest being allowed on the sum borrowed for the whole time, and on each payment from the time it is made to the end of the 60 days, at the rate of 10 per cent. for 12 months of 30 days each? *Ans.* $\$1\frac{61}{7259}$.

7. There are four numbers in arithmetical progression; the sum of the squares of the extremes is 65, and the sum of the squares of the means is 61. Required the numbers.

Ans. 4, 5, 6, 7.

8. The sum of four numbers in arithmetical progression is 24, and their continued product is 945; what are the numbers?

Ans. 3, 5, 7, 9.

9. A certain number consists of three digits, which are in arithmetical progression; if the number be divided by the sum of its digits the quotient will be 26, and if 198 be added to the number the order of its digits will be inverted. What is the number? *Ans.* 234.

10. From two towns which were 102 miles apart, two persons, A and B, set out to meet each other; A went 3 miles the first day, 5 the next, 7 the next, and so on; B went 4 miles the first day, 6 the next, 8 the next, and so on. In how many days did they meet? *Ans.* 6.

11. A quantity of corn is to be divided among 21 persons, and is calculated to last a certain time if each of them receive a peck every week; during the distribution it is found that one person dies at the end of every week, and then the corn lasts twice as long as was expected lacking one week. Find the quantity of corn. *Ans.* 231 pecks.

GEOMETRICAL PROGRESSION.

356. A ***Geometrical Progression*** is a series of quantities, each of which is equal to the preceding one multiplied by a constant factor.

357. The constant factor is called the *ratio;* and if the first term is positive, the progression will be an *increasing,* or a *decreasing* series, according as the ratio is greater or less, than unity. Thus,

$$2,\ 6,\ 18,\ 54,\ 162, \ldots.$$

is an increasing geometrical series, in which the ratio is 3; and

$$81,\ 27,\ 9,\ 3,\ 1,\ \tfrac{1}{3},\ \tfrac{1}{9}, \ldots.$$

is a decreasing geometrical series, in which the ratio is $\frac{1}{3}$.

358. When a geometrical progression is supposed to terminate, the first and last terms are called *extremes,* and all the terms between the first and last are called *geometrical means.*

359. To find the last term of a geometrical progression.

Let a denote the first term, r the ratio, l the last term, and n the number of terms. Then the series will be represented thus:

$$a,\ ar,\ ar^2,\ ar^3, \ldots.\ l.$$

Now we perceive that in any term the exponent of r is equal to the number of preceding terms. Hence, we shall have

$$l = a^{n-1} \quad \ldots\ldots\ldots \quad (A).$$

360. To find the sum of the terms in a geometrical progression.

Denote the sum of the series by S; then

$$S = a + ar + ar^2 + ar^3 \ldots. + ar^{n-1} \quad \ldots \quad (1),$$

and

$$rS = \quad ar + ar^2 + ar^3 \ldots. + ar^{n-1} + ar^n \quad (2).$$

Hence by subtraction, remembering that $ar^n = rl$,

$$rS - S = ar^n - a \quad \ldots\ldots \quad (3),$$

or

$$rS = S = rl - a \quad \ldots\ldots \quad (4).$$

Thus we obtain two expressions for S, as follows:

$$S = \frac{a(r^n - 1)}{r - 1} \quad \ldots\ldots\ldots \quad (B),$$

$$S = \frac{rl - a}{r - 1} \quad \ldots\ldots\ldots \quad (B').$$

361. To find the sum of a decreasing geometrical series, when the number of terms is infinite.

By changing signs in the numerator and denominator, equation (B) may be written

$$S = \frac{a(1 - r^n)}{1 - r}.$$

Now suppose r less than unity; then the larger n is, the smaller will r^n be; and by making n large enough, r^n may be made less than any assignable quantity, or zero. Hence, if the number of terms is infinite, r^n may be neglected in comparison with unity; and we shall have as the formula for an infinite series,

$$S = \frac{a}{1 - r} \quad \ldots\ldots\ldots \quad (C).$$

362. To insert a given number of geometrical means between two given quantities.

Let n' denote the number of means to be inserted; then the whole series will consist of $n' + 2$ terms. Hence, putting $n = n' + 2$ in equation (A),

$$l = ar^{n'+1};$$

whence

$$r = \sqrt[n'+1]{\frac{l}{a}} \quad \ldots\ldots\ldots \quad (D).$$

Having found the ratio, the required means may be obtained.

363. Since the terms of a geometrical series, taken consecutively, have the same ratio one to another, it follows that they are in continued proportion (**327**). Hence,

1st. *When three terms are in geometrical progression, the product of the extremes is equal to the square of the mean.*

2d. *When four terms are in geometrical progression, the product of the means is equal to the product of the extremes.*

APPLICATION OF THE FORMULAS.

364. The two equations,

$$l = ar^{n-1}, \qquad S = \frac{a(r^n - 1)}{r - 1},$$

contain the five quantities, a, r, l, n, S, any three of which being

given, the other two may be found; for, by substitution of the given values, the result will always be two equations involving but two unknown quantities.

In this general problem there will be ten cases, as in the corresponding problem of Arithmetical Progression. We cannot, however, obtain a solution of all the cases, by simple or quadratic equations.

1st. The quantity n enters the two equations only as an *exponent*, and its value cannot be obtained by the common methods of solving an equation. The process involves the principle of logarithms, and will be presented in its proper place.

2d. The quantity r is affected by an exponent in both equations; and its value must be obtained by extracting the $(n-1)th$ or the n^{th} root of a quantity. When n is not large, r can readily be found by inspection or trial.

3d. The values of a, l, and S may be found by means of simple equations, as in Arithmetical Progression.

1. The first term of a geometrical progression is 3, and the ratio 2; find the 12th term, and the sum of the series.

We have given,

$$a = 3, \quad r = 2, \quad n = 12.$$

Whence by formulas (A) and (B),

$$\left.\begin{aligned} l &= 3 \times 2^{11} = 3 \times 2048 = 6144 \\ S &= \frac{3\,(2^{12}-1)}{2-1} = 3 \times 4095 = 12285 \end{aligned}\right\}, \textit{ Ans.}$$

2. The sum of a geometrical progression is 1820, the number of terms 6, and the ratio 3; find the first term, and the last term.

We have given,

$$S = 1820, \quad n = 6, \quad r = 3.$$

By formula (B),

$$1820 = \frac{a\,(3^6-1)}{3-1} = 364a;$$

$$a = 5, \text{ first term.}$$

Then by formula (A),

$$l = 5 \times 3^5 = 1215, \text{ last term.}$$

3. It is required to find 3 geometrical means between 6 and 486. By formula (D),

$$r = \sqrt[4]{\tfrac{486}{6}} = \sqrt[4]{81} = 3.$$

Therefore, the series is 6, 18, 54, 162, 486, *Ans.*

4. Find the sum of the series 6, 2, $\frac{2}{3}$, $\frac{2}{9}$ .... to infinity.

We have given, $a = 6$, $r = \frac{1}{3}$; hence, by formula (C),

$$S = \frac{6}{1 - \frac{1}{3}} = 9, \textit{ Ans.}$$

5. Find the exact value of the decimal .454545 .... to infinity.

This is a circulating decimal, and may be expressed thus:

$$\frac{45}{100} + \frac{45}{10000} + \frac{45}{1000000} + \text{etc.}$$

In all such cases, the repetend, taken with its local value, will be the first term of a geometrical series, of which the ratio will be $\frac{1}{10}$ or some power of $\frac{1}{10}$. In the present example we have

$$a = \frac{45}{100}, \quad r = \frac{1}{100}; \qquad \text{hence,}$$

$$S = \frac{45}{100} \div \left(1 - \frac{1}{100}\right) = \frac{45}{100} \times \frac{100}{99} = \frac{5}{11}, \textit{ Ans.}$$

6. Find the value of $1 - \frac{x}{a} + \frac{x^2}{a^2} - \frac{x^3}{a^3} + \ldots$ to infinity.

We have $a = 1$, $r = -\frac{x}{a}$; hence,

$$S = \frac{1}{1 + \frac{x}{a}} = \frac{a}{a + x}, \textit{ Ans.}$$

EXAMPLES FOR PRACTICE.

1. Find the sum of 9 terms of the series 1, 2, 4, 8, ...
Ans. 511.

2. Find the 8th term of the progression 2, 6, 18, 54, ...
Ans. 4374.

3. Find the sum of 10 terms of the series 1, $\frac{2}{3}$, $\frac{4}{9}$, $\frac{8}{27}$, ...
Ans. $\frac{174075}{59049}$.

4. Find two geometrical means between 24 and 192.

Ans. 48, 96.

5. Find 7 geometrical means between 3 and 768.

Ans. 6, 12, 24, 48, 96, 192, 384.

6. Find the value of $1 + \frac{3}{4} + \frac{9}{16} + \frac{27}{64} + \ldots\ldots$ to infinity.

Ans. 4.

7. Find the value of $\frac{5}{3} + 1 + \frac{3}{5} + \frac{9}{25} + \ldots\ldots$ to infinity.

Ans. $4\frac{1}{6}$.

8. Find the value of $5 + \frac{5}{3} + \frac{5}{9} + \frac{5}{27} + \ldots\ldots$ to infinity.

Ans. $7\frac{1}{2}$.

9. Find the value of the decimal .323232 .... to infinity.

Ans. $\frac{32}{99}$.

10. Find the value of the decimal .212121 .... to infinity.

Ans. $\frac{7}{33}$.

11. Find the value of $\frac{1}{2} - \frac{1}{4} + \frac{1}{8} - \frac{1}{16} + \frac{1}{32} - \ldots\ldots$ to infinity.

Ans. $\frac{1}{3}$.

12. Find the value of $\frac{1}{5} - \frac{1}{25} + \frac{1}{125} - \frac{1}{625} + \ldots\ldots$ to infinity.

Ans. $\frac{1}{6}$.

13. Find the value of $1 + \frac{x}{a} + \frac{x^2}{a^2} + \frac{x^3}{a^3} + \ldots\ldots$ to infinity.

Ans. $\frac{a}{a - x}$.

14. Find the value of $\frac{1}{a} - \frac{x^2}{a^3} + \frac{x^4}{a^5} - \frac{x^6}{a^7} + \ldots\ldots$ to infinity.

Ans. $\frac{a}{a^2 + x^2}$.

15. The sum of a geometrical series is 1785, the ratio 2, and the number of terms 8; find the first term. *Ans.* 7.

16. The sum of a geometrical series is 7812, the ratio 5, and the number of terms 6; find the last term. *Ans.* 6250.

17. The first term of a geometrical series is 5, the last term 1215, and the number of terms 6. What is the ratio? *Ans.* 3.

18. A man purchased a house with ten doors, giving $1 for the first door, $2 for the second, $4 for the third, and so on. What did the house cost him? *Ans.* $1023.

PROBLEMS IN GEOMETRICAL PROGRESSION

TO WHICH THE FORMULAS DO NOT IMMEDIATELY APPLY.

365. The terms of a geometrical progression are represented in a general manner as follows:

$$x, \quad xy, \quad xy^2, \quad xy^3, \ldots.$$

In the solution of problems, however, the following notation is generally preferable:

1st. When the number of terms is *odd*, the series may be represented thus:

$$x^2, \quad xy, \quad y^2;$$

$$\frac{x^3}{y}, \quad x^2, \quad xy, \quad y^2, \quad \frac{y^3}{x};$$

2d. When the number of terms is *even*, the series may be expressed thus:

$$\frac{x^2}{y}, \quad x, \quad y, \quad \frac{y^2}{x};$$

$$\frac{x^3}{y^2}, \quad \frac{x^2}{y}, \quad x, \quad y, \quad \frac{y^2}{x}, \quad \frac{y^3}{x^2}.$$

We may also represent three terms as follows:

$$x, \quad \sqrt{xy}, \quad y.$$

1. The sum of three numbers in geometrical progression is 26, and the sum of their squares 364. What are the numbers?

Let the numbers be denoted by x, $\sqrt{xy}$, y.

Then $\qquad x + \sqrt{xy} + y = 26 = a \quad \ldots\ldots \quad (1),$

and $\qquad x^2 + xy + y^2 = 364 = b \quad \ldots\ldots \quad (2).$

Transposing $\sqrt{xy}$ in (1), squaring and reducing,

$$x^2 + xy + y^2 = a^2 - 2a\sqrt{xy} \quad \ldots \quad (3).$$

From (2) and (3), $\qquad a^2 - 2a\sqrt{xy} = b;$

whence, $\qquad \sqrt{xy} = \frac{a^2 - b}{2a} = 6.$

From (1) and (2), $\qquad x = 2, \quad \text{and} \quad y = 18.$

Hence, the numbers are, 2, 6, 18, *Ans.*

2. The sum of four numbers in geometrical progression is 15 or a, and the sum of their squares 85 or b. What are the numbers?

Taking the proper notation for an *even* number of terms, we have

$$\frac{x^2}{y} + x + y + \frac{y^2}{x} = a \quad . \quad . \quad . \quad . \quad (1),$$

and

$$\frac{x^4}{y^2} + x^2 + y^2 + \frac{y^4}{x^2} = b \quad . \quad . \quad . \quad . \quad (2).$$

Assume $x + y = s$, and $xy = p$; then by **301**,

$$x^2 + y^2 = s^2 - 2p, \qquad x^3 + y^3 = s^3 - 3sp.$$

Substituting the values of $(x + y)$ and $(x^2 + y^2)$, in (1) and (2),

$$\frac{x^2}{y} + \frac{y^2}{x} = a - s \quad . \quad . \quad . \quad . \quad . \quad (3),$$

$$\frac{x^4}{y^2} + \frac{y^4}{x^2} = b - s^2 + 2p \quad . \quad . \quad . \quad (4).$$

Squaring (3), and then transposing $2xy$, or $2p$,

$$\frac{x^4}{y^2} + \frac{y^4}{x^2} = (a - s)^2 - 2p \quad . \quad . \quad . \quad (5);$$

whence, from (4) and (5), $(a - s)^2 - 2p = b - s^2 + 2p$;

or, $$a^2 - 2as + 2s^2 - 4p = b \quad . \quad . \quad . \quad (6).$$

Clearing (3) of fractions, and putting $xy = p$ in second member,

$$x^3 + y^3 = ap - ps; \quad \text{or,} \quad s^3 - 3sp = ap - ps$$

whence, $$p = \frac{s^3}{a + 2s} \quad . \quad . \quad . \quad . \quad . \quad (7).$$

Substituting this value of p in (6), and reducing,

$$a^3 - 2as^2 = ab + 2bs;$$

or, $$as^2 + bs = \frac{a}{2}(a^2 - b).$$

Restoring the numerical values of a and b,

$$15s^2 + 85s = 70 \times 15,$$

whence, $$s = 6.$$

Substituting the values of a and s in (7), we obtain

$$p = 8;$$

that is, $$x + y = 6, \qquad xy = 8,$$

whence, $$x = 2, \qquad y = 4.$$

Therefore, the required numbers are 1, 2, 4, 8, *Ans.*

3. There are three numbers in geometrical progression; their sum is 21, and the sum of their squares is 189. Find the numbers. *Ans.* 3, 6, 12.

4. Divide the number 210 into three parts, so that the last shall exceed the first by 90, and the parts be in geometrical progression. *Ans.* 30, 60, and 120.

5. The sum of four numbers in geometrical progression is 30; and the last term divided by the sum of the mean terms gives $1\frac{1}{3}$. What are the numbers? *Ans.* 2, 4, 8, and 16.

6. The sum of the first and third of four numbers in geometrical progression is 148, and the sum of the second and fourth is 888. What are the numbers? *Ans.* 4, 24, 144, and 864.

7. It is required to find three numbers in geometrical progression, such that their sum shall be 14, and the sum of their squares 84. *Ans.* 2, 4, and 8.

8. There are four numbers in geometrical progression, the second of which is less than the fourth by 24; and the sum of the extremes is to the sum of the means as 7 to 3. What are the numbers? *Ans.* 1, 3, 9, and 27.

9. There are three numbers in geometrical progression; the sum of the first and second is 20, and the difference of the second and third is 30. What are the numbers? *Ans.* 5, 15, 45.

10. The continued product of three numbers in geometrical progression is 216, and the sum of the squares of the extremes is 328. What are the numbers? *Ans.* 2, 6, 18.

11. The sum of three numbers in geometrical progression is 13, and the sum of the extremes being multiplied by the mean, the product is 30. What are the numbers? *Ans.* 1, 3, and 9.

12. There are three numbers in geometrical progression; their continued product is 64, and the sum of their cubes is 584. What are the numbers? *Ans.* 2, 4, 8.

13. There are three numbers in geometrical progression; their continued product is 1, and the difference of the first and second is to the difference of the second and third as 5 to 1. What are the numbers? *Ans.* 5, 1, $\frac{1}{5}$.

14. The sum of 120 dollars was divided among four persons in such a manner that the shares were in arithmetical progression; if the second and third had each received 12 dollars less, and the fourth 24 dollars more, the shares would have been in geometrical progression. Find the shares. *Ans.* \$3, \$21, \$39, and \$57.

15. There are three numbers in geometrical progression, whose sum is 31, and the sum of the first and last is 26. What are the numbers? *Ans.* 1, 5, and 25.

16. The sum of six numbers in geometrical progression is 189, and the sum of the second and fifth is 54. What are the numbers? *Ans.* 3, 6, 12, 24, 48, and 96.

17. The sum of six numbers in geometrical progression is 189, and the sum of the two means is 36. What are the numbers? *Ans.* 3, 6, 12, 24, 48, and 96.

18. A man borrowed p dollars; what sum must he pay yearly in order to cancel the debt in n years, interest being allowed on the unpaid parts of the principal at r cents per annum on a dollar? *Ans.* $\frac{pr(1+r)^n}{(1+r)^n-1}$ dollars.

IDENTICAL EQUATIONS.

366. An ***Identical Equation*** is one in which the two members are either the same algebraic expression, or the one member is merely another form for the other. In every case, either the one member may be reduced to the other directly, or the two members may be reduced to some expression different from either, from which both members may be supposed to originate. Thus,

$$ax + b = ax + b,$$

$$x^2 + (a - b)x - ab = (x + a)(x - b),$$

$$1 - x + x^2 - \frac{x^3}{1 + x} = \frac{1}{x} - \frac{1}{x^2} + \frac{1}{x^2(1 + x)},$$

are identical equations. In the first, the two members have

exactly the same form. In the second, the second member may be reduced to the form of the first, by performing the multiplication indicated. In the third, each member may be reduced to the fraction, $\frac{1}{1+x}$.

367. There are certain properties of identical equations, which are of great importance in the further treatment of series, and in the general theory of equations.

In order to investigate these properties, let us first consider what any term containing the variable x, as ax^n, will become when $x = 0$, under the various conditions of the exponent.

1. Suppose n to be positive; then if $x = 0$, we have

$$ax^n = a \cdot 0^n = 0.$$

2. Suppose n to be negative; then if $x = 0$, we have

$$ax^{-n} = \frac{a}{x^n} = \frac{a}{0} = \infty.$$

3. Suppose n to be nothing or zero; then if $x = 0$, we have

$$ax^n = a \cdot 0^0 = a \cdot 1 = a.$$

368. We are now prepared to demonstrate the following propositions:

I. *An identical equation is satisfied for any value whatever of the unknown quantity.*

The truth of this proposition follows directly from the definition of an identical equation. It is implied in all algebraic transformations, that the value of a function is not changed by changing its form, whatever quantities the symbols represent. Hence, if the two members of an equation are the same in form, or reducible to the same expression, they must be equal, whatever value be substituted for the unknown quantity.

To illustrate this principle, we will take the following identical equation.

$$\left\{(x-3)^2 + 1 + (x-2)^2\right\}^2 \doteq 2\left\{(x-3)^4 + 1 + (x-2)^4\right\},$$

where the form is such that the identity of the two members is not apparent from inspection.

If in this equation we make x equal to 1, 2, 3, 4, 5, etc., successively, we shall have,

$$\begin{aligned}
\{4+1+1\}^2 &= 2\{16+1+\ \ 1\},\\
\{1+1+0\}^2 &= 2\{\ \ 1+1+\ \ 0\},\\
\{0+1+1\}^2 &= 2\{\ \ 0+1+\ \ 1\},\\
\{1+1+4\}^2 &= 2\{\ \ 1+1+16\},\\
\{4+1+9\}^2 &= 2\{16+1+81\}, \text{ etc.,}
\end{aligned}$$

every result being a true equation.

II. Conversely: *Every equation which is satisfied for any value whatever of the unknown quantity, is an identical equation.*

Suppose the given equation to be cleared of fractions, and each member arranged according to the ascending powers of the unknown quantity. Then the equation may be represented thus:

$$Ax^a + Bx^b + Cx^c + \ldots\ldots = A'x^{a'} + B'x^{b'} + C'x^{c'} + \ldots\ldots \quad (1),$$

in which, by hypothesis, we have

$$a < b < c \ldots\ldots, \text{ and } a' < b' < c' \ldots\ldots$$

It is implied, also, that the coefficients, A, B, C, etc., and A', B', C', etc., *are all finite quantities greater than zero, and independent of* x; and the number of terms may be limited or unlimited.

Divide both members of equation (1) by x^a; we have

$$A + Bx^{b-a} + Cx^{c-a} + \ldots\ldots = A'x^{a'-a} + B'x^{b'-a} + C'x^{c'-a} + \ldots\ldots \quad (2),$$

in which the exponents, $b-a$, $c-a$, etc., in the first member, are all positive, because $a < b < c \ldots\ldots$

Now by hypothesis, the given equation is true for all values of x; hence every modification of it will be true for all values of x. Make $x = 0$; then in the first member of equation (2), every term after the first will reduce to zero (**367**, 1), and we shall have

$$A = A'x^{a'-a} + B'x^{b'-a} + C'x^{c'-a} + \ldots\ldots \quad (3).$$

Now since $\qquad a' < b' < c' \ldots\ldots,$

we must have $(a'-a) < (b'-a) < (c'-a) < \ldots\ldots$

Hence, in equation (3), the first exponent, $a'-a$, is the *least of all.* But we observe,

1st. The exponent, $a'-a$, cannot be a positive quantity; for

in that case the term containing it would reduce to zero when $x = 0$ (**367**, 1), and we should have $A = 0$, which is contrary to the implied conditions of the proposition.

2d. The exponent, $a' - a$, cannot be a negative quantity; for in that case the term containing it would reduce to infinity when $x = 0$ (**367**, 2), and we should have $A = \infty$, which is also contrary to the implied conditions of the proposition. Now since $a' - a$ can neither be a positive nor a negative quantity, it must be nothing or zero; that is,

$$a' - a = 0, \quad \text{or} \quad a' = a.$$

It follows also that each of the other exponents, $b' - a$, $c' - a$, etc., in equation (3), is positive, being algebraically greater than zero; hence all the terms after the first in the second member of this equation, must disappear when $x = 0$ (**367**, 1), and we shall have

$$A = A'x^0 = A'.$$

Now since A and A' are independent of x, we shall have

$$Ax^a = A'x^{a'},$$

whatever be the value of x. We may therefore suppress these terms in equation (1). There will result

$$Bx^b + Cx^c + \ldots\ldots = B'x^{b'} + C'x^{c'} + \ldots\ldots,$$

whence, by reasoning as before, we shall find that

$$b = b', \quad B = B'$$
$$c = c', \quad C = C', \text{ etc.}$$

That is, equation (1) is an identical equation, the two members having the *same form*. Hence, the given equation is also identical, and the proposition is proved.

It is obvious that the preceding demonstration will apply if one or more of the exponents, $a, b, c, \ldots\ldots$ are negative; or if $a = 0$, in which case each member will contain an absolute term.

III. *In every equation which is satisfied for any value whatever of the unknown quantity, and which involves like powers of this quantity in the two members, the coefficients of the corresponding powers will be equal, each to each.*

Let us assume the equation,

$$Ax^a + Bx^b + Cx^c + \ldots\ldots = A'x^a + B'x^b + C'x^c + \ldots\ldots,$$

the number of terms being either limited or unlimited.

Now if this equation is capable of being satisfied for every value of x, then according to the preceding demonstration, not only must the exponents of x in the two members be equal respectively, but the coefficients also must be equal, each to each; that is,

$$A = A', \quad B = B', \quad C = C', \text{ etc.}$$

Every such equation is obviously identical, though it is not necessary that A, B, C, etc., should be of the same *form* respectively, as A', B', C', etc.

IV. *In every equation which is satisfied for any value whatever of the unknown quantity, and which has zero for one of its members, the coefficients of the different powers of the unknown quantity are separately equal to zero.*

Let

$$Ax^a + Bx^b + Cx^c + Dx^d + \ldots\ldots = 0 \quad \ldots\ldots (1),$$

represent the equation, arranged according to the ascending powers of x. The coefficients, A, B, C, D, etc., are supposed to be independent of x, and consequently the same for all values of x.

Divide every term in this equation by x^a; we shall have

$$A + Bx^{b-a} + Cx^{c-a} + Dx^{d-a} + \ldots\ldots = 0 \quad \ldots (2).$$

In this equation make $x = 0$; then since the exponents, $b - a$, $c - a$, $d - a$, etc., are all positive, every term after the first will reduce to zero (**367**, 1), and we shall have

$$A = 0.$$

Suppressing Ax^a in equation (1), and then dividing through by x^b, we obtain

$$B + Cx^{c-b} + Dx^{d-b} + \ldots\ldots = 0 \quad \ldots\ldots (3).$$

In this equation make $x = 0$, and we have

$$B = 0$$

In like manner we may prove that each of the other coefficients is equal to zero.

It is important to observe in this connection that the coefficients, A, B, C, D, etc., must be supposed to represent *polynomial expressions, which reduce to zero in consequence of having positive and negative parts that are respectively equal to each other.*

DECOMPOSITION OF RATIONAL FRACTIONS.

369. By means of the properties of identical equations, a fraction may often be separated into two or more partial fractions, whose denominators shall be simpler than the given denominator. In every such case, the given fraction is the sum of the partial fractions; hence its denominator will be a common multiple of the denominators of the partial fractions.

1. Separate $\frac{8x-31}{x^2-7x+10}$ into partial fractions.

By inspection, we perceive that

$$x^2-7x+10=(x-5)(x-2).$$

Now assume

$$\frac{8x-31}{(x-5)(x-2)}=\frac{A}{x-5}+\frac{B}{x-2} \quad . \; . \; . \; (1).$$

Since the first member is simply the sum of the two fractions in the second member, this is obviously an *identical equation.* Clearing of fractions and uniting terms,

$$8x-31=(A+B)x-(2A+5B) \; . \; . \; . \; (2),$$

in which 31 in the first member, and $(2A+5B)$ in the second, may be considered as coefficients of x^0. Now according to **368,** III, the coefficients of the like powers of x in the two members must be equal; therefore,

$$A+B=8 \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; (3),$$
$$2A+5B=31 \; . \; . \; . \; . \; . \; . \; . \; . \; . \; (4).$$

From these equations we readily obtain

$$A=3, \text{ and } B=5;$$

whence, from equation (1), we have

$$\frac{8x-31}{x^2-7x+10}=\frac{3}{x-5}+\frac{5}{x-2}, \; Ans.$$

It should be observed that equations (3) and (4) are the *equations of condition,* which must exist in order that equation (1) shall be true for all values of x.

2. Separate $\frac{7x^2+x}{(x+1)(2x-1)}$ into partial fractions.

Suppose, if possible,

$$\frac{7x^2+x}{(x+1)(2x-1)}=\frac{A}{x+1}+\frac{B}{2x-1} \quad . . . (1);$$

clearing of fractions,

$$7x^2+x=(2A+B)x+(B-A);$$

transposing all the terms to the first member,

$$7x^2+(1-2A-B)x+(A-B)=0 \quad . . (2).$$

If this equation be possible, it must be an identical equation; and as one member is zero, the coefficients of the different powers of x must be separately equal to zero (**368**, IV); and we shall have

$$7=0,$$

which is absurd. Hence, we infer that the fraction cannot be separated into partial fractions, *having numerators independent of* **x**.

Again, assume

$$\frac{7x^2+x}{(x+1)(2x-1)}=\frac{Ax}{x+1}+\frac{Bx}{2x-1} \quad . . . (1);$$

clearing of fractions and collecting terms,

$$7x^2+x=(2A+B)x^2+(B-A)x \quad . . . (2);$$

equating the coefficients of like powers of x,

$$2A+B=7,$$
$$B-A=1;$$

whence, $\quad A=2,\ B=3;$

and by substitution in equation (1),

$$\frac{7x^2+x}{(x+1)(2x-1)}=\frac{2x}{x+1}+\frac{3x}{2x-1}, \text{ Ans.}$$

From this example we learn that if we assume an impossible form for the partial fractions, the fact will be made apparent by some absurdity in the equations of condition.

NOTE.—If the given denominator consists of three or more factors, there will be three or more partial fractions. But there will always be as many equations of condition as there are numerators to be determined.

EXAMPLES FOR PRACTICE.

1. Resolve $\frac{7x-24}{x^2-9x+14}$ into partial fractions.

$$Ans.\ \frac{5}{x-7}+\frac{2}{x-2}.$$

2. Resolve $\frac{20x+2}{2x^2+3x-20}$ into partial fractions.

$$Ans.\ \frac{8}{2x-5}+\frac{6}{x+4}.$$

3. Resolve $\frac{6x^2-22x+18}{(x-1)(x^2-5x+6)}$ into partial fractions.

$$Ans.\ \frac{1}{x-1}+\frac{2}{x-2}+\frac{3}{x-3}.$$

4. Resolve $\frac{x+2}{x^3-x}$ into partial fractions.

$$Ans.\ \frac{1}{2(x+1)}+\frac{3}{2(x-1)}-\frac{2}{x}.$$

5. Resolve $\frac{10}{x^4-13x^2+36}$ into partial fractions.

$$Ans.\ \frac{1}{2(x+2)}-\frac{1}{2(x-2)}-\frac{1}{3(x+3)}+\frac{1}{3(x-3)}.$$

THE RESIDUAL FORMULA.

370. It has been shown (**89,** 4) that x^m-y^m is exactly divisible by $x-y$, if m is a positive whole number. The form of the quotient is as follows:

$$\frac{x^m-y^m}{x-y}=x^{m-1}+x^{m-2}y+x^{m-3}y^2+x^{m-4}y^3+\ldots+y^{m-1},$$

the number of terms in the quotient being equal to m.

Now suppose $x=y$; then each term will become x^{m-1}, and since there are m terms, we have the formula,

$$\left(\frac{x^m-y^m}{x-y}\right)_{y=x}=mx^{m-1}\quad.\ .\ .\ .\ .\ .\quad(A).$$

The subscript equation, $y=x$, is used to indicate the condition under which the first member of (A) will be equal to the second.

371. We will now show that this formula is true, whatever be the value of m. There will be two cases:

1st. *When* m *is positive and fractional.*

Assume $m = \frac{r}{s}$; then $x^m - y^m = x^{\frac{r}{s}} - y^{\frac{r}{s}}$.

Let $x^{\frac{1}{s}} = z$; then $x^{\frac{r}{s}} = z^r$, and $x = z^s$.

Also let $y^{\frac{1}{s}} = u$; then $y^{\frac{r}{s}} = u^r$, and $y = u^s$.

By proper substitutions we have

$$\frac{x^{\frac{r}{s}} - y^{\frac{r}{s}}}{x - y} = \frac{z^r - u^r}{z^s - u^s} = \frac{\frac{z^r - u^r}{z - u}}{\frac{z^s - u^s}{z - u}} \quad . \ . \ . \quad (1).$$

Now suppose $x = y$, then $z = u$; and since r and s are *positive whole numbers,* we have from (1),

$$\left\{\frac{x^{\frac{r}{s}} - y^{\frac{r}{s}}}{x - y}\right\}_{y=x} = \frac{\left(\frac{z^r - u^r}{z - u}\right)_{u=z}}{\left(\frac{z^s - u^s}{z - u}\right)_{u=z}} = \frac{rz^{r-1}}{sz^{s-1}} = \frac{r}{s}z^{r-s} = \frac{r}{s}x^{\frac{r}{s}-1}.$$

Hence, the formula is true when the exponent is positive and fractional.

2d. *When* m *is negative, and either integral or fractional.*

Suppose the exponent of x and y to be $-m$; we shall have

$$x^{-m} - y^{-m} = -x^{-m}y^{-m}(x^m - y^m); \qquad \text{hence,}$$

$$\frac{x^{-m} - y^{-m}}{x - y} = -x^{-m}y^{-m}\left(\frac{x^m - y^m}{x - y}\right) \quad . \ . \ . \quad (1).$$

Now suppose $x = y$; then, whether m be integral or fractional, we shall have, from the principles already established,

$$-x^{-m}y^{-m} = -x^{-2m}, \quad \text{and} \quad \frac{x^m - y^m}{x - y} = mx^{m-1}; \qquad \text{hence,}$$

$$\left(\frac{x^{-m} - y^{-m}}{x - y}\right)_{y=x} = (-x^{-2m}) \times (mx^{m-1}) = -mx^{-m-1}.$$

Hence, the formula holds true universally.

BINOMIAL THEOREM.

372. The ***Binomial Theorem*** has for its object the development of a binomial with any exponent, into a series. This theorem is expressed by an equation, called the Binomial Formula.

373. *It is required to expand* $(a+x)^n$ *into a series,* n *being any real quantity, positive or negative, entire or fractional.*

We observe that

$$a + x = a\left(1 + \frac{x}{a}\right); \text{ therefore } (a + x)^n = a^n\left(1 + \frac{x}{a}\right)^n.$$

Hence, if we first expand $\left(1 + \frac{x}{a}\right)^n$, and then multiply the result by a^n, we shall have the expansion of $(a + x)^n$.

$$\text{Put } z = \frac{x}{a}; \text{ then } \left(1 + \frac{x}{a}\right)^n = (1 + z)^n.$$

Let us now assume the equation,

$$(1 + z)^n = A + Bz + Cz^2 + Dz^3 + Ez^4 + \ldots\ldots \quad (1),$$

in which A, B, C, D, etc., are independent of z. We are to find the values of these coefficients which will render equation (1) true for all possible values of z.

Suppose $z = 0$; then from equation (1), we have $A = 1$.

Hence, the assumed development becomes

$$(1 + z)^n = 1 + Bz + Cz^2 + Dz^3 + Ez^4 + \ldots\ldots \quad (2)$$

for all values of z. Put $z = u$; then

$$(1 + u)^n = 1 + Bu + Cu^2 + Du^3 + Eu^4 + \ldots\ldots \quad (3).$$

Subtracting (3) from (2), and dividing the result by $z - u$,

$$\frac{(1+z)^n - (1+u^n)}{z-u} = B + C\left(\frac{z^2-u^2}{z-u}\right) + D\left(\frac{z^3-u^3}{z-u}\right) + \ldots\ldots \quad (4).$$

Let $P = 1 + z$, and $Q = 1 + u$; then $P - Q = z - u$.

Equation (4) now becomes

$$\frac{P^n - Q^n}{P - Q} = B + C\left(\frac{z^2-u^2}{z-u}\right) + D\left(\frac{z^3-u^3}{z-u}\right) + E\left(\frac{z^4-u^4}{z-u}\right) + \ldots\ldots \quad (5).$$

Now suppose $z = u$; then $P = Q$. And by the Residual Formula (**370**), we shall have

$$\left(\frac{P^n-Q^n}{P-Q}\right)_{Q=P} = nP^{n-1} = n\,(1+z)^{n-1}.$$

$$\text{Also}\quad \begin{cases} \left(\dfrac{z^2-u^2}{z-u}\right)_{u=z} = 2z\,; \\ \left(\dfrac{z^3-u^3}{z-u}\right)_{u=z} = 3z^2\,; \\ \left(\dfrac{z^4-u^4}{z-u}\right)_{u=z} = 4z^3, \text{ etc.} \end{cases}$$

Substituting these values in equation (5),

$$n\,(1+z)^{n-1} = B + 2Cz + 3Dz^2 + 4Ez^3 + \ldots . \quad . \quad . \quad . \quad (6).$$

Multiplying both members of equation (6) by $(1+z)$,

$$n\,(1+z)^n = B + \left.\begin{matrix}2C \\ +\ B\end{matrix}\right| z + \left.\begin{matrix}3D \\ +2C\end{matrix}\right| z^2 + \left.\begin{matrix}4E \\ +3D\end{matrix}\right| z^3 + \ldots . \quad . \quad . \quad . \quad (7).$$

Multiplying both members of equation (2) by n,

$$n\,(1+z)^n = n + nBz + nCz^2 + nDz^3 + \ldots . \quad . \quad . \quad . \quad (8).$$

Now by equating the second members of (7) and (8) we shall have an *identical equation*, because it may be satisfied for any value of z. Therefore the coefficients of the like powers of z in (7) and (8) are equal, each to each (**368**, III), and we shall have

$$B = n\,;$$

$$2C + B = nB, \quad \text{or} \quad C = B\left(\frac{n-1}{2}\right);$$

$$3D + 2C = nC, \quad \text{or} \quad D = C\left(\frac{n-2}{3}\right);$$

$$4E + 3D = nD, \quad \text{or} \quad E = D\left(\frac{n-3}{4}\right); \text{ etc.}$$

Therefore, the values of the coefficients are

$$A = 1\,;$$

$$B = n.$$

$$C = \frac{n\,(n-1)}{2};$$

$$D = \frac{n\,(n-1)\,(n-2)}{2 \cdot 3};$$

$$E = \frac{n\,(n-1)\,(n-2)\,(n-3)}{2 \cdot 3 \cdot 4}; \text{ etc.}$$

Substituting these values in (1),

$$(1+z)^n = 1+nz+\frac{n(n-1)}{2}z^2+\frac{n(n-1)(n-2)}{2\ .\ 3}z^3+\dots\ .\ .\ .\ (a);$$

and by restoring the value of z, which is $\frac{x}{a}$,

$$\left(1+\frac{x}{a}\right)^n = 1+n\frac{x}{a}+\frac{n(n-1)}{2}\cdot\frac{x^2}{a^2}+\frac{n(n-1)(n-2)}{2\ .\ 3}\cdot\frac{x^3}{a^3}+\dots\ .\ .\ .\ (b);$$

or, finally, multiplying both members of (b) by a^n,

$$(a+x)^n = a^n+na^{n-1}x+\frac{n(n-1)}{2}a^{n-2}x^2+\frac{n(n-1)(n-2)}{2\ .\ 3}a^{n-3}x^3+\dots\ .\ .\ .\ (c).$$

Equation (c) is the binomial formula, as it is usually written. It will be observed, however, that in the three equations, (a), (b), (c), the coefficients, or the factors depending on n, are the same; and in practice, either (a), (b), or (c) may be employed, according to the form of the binomial to be expanded.

374. By inspecting the general formula (c), we perceive that in the expansion of a binomial in the form of $a+x$, the law of the exponents is as follows:

1. *The exponents of the leading letter in the successive terms form a series, commencing in the first term with the exponent of the binomial, and diminishing by* 1 *to the right.*

2. *The exponents of the second letter form a series, commencing in the second term with unity, and increasing by* 1 *to the right.*

And the law of the coefficients is as follows:

3. *The coefficient of the first term is unity, and that of the second term is the exponent of the required power.*

4. *If the coefficient of any term be multiplied by the exponent of the leading letter in that term; and divided by the exponent of the second letter plus* 1, *the result will be the coefficient of the following term.*

375. If we take the *least factor* in each of the successive coefficients of the expansion, commencing at the *second*, we have a decreasing series,

$$n,\quad (n-1),\quad (n-2),\quad (n-3),\quad \text{etc.},$$

in which the common difference is unity.

Suppose n to be a positive integer, then the least factor in the numerator in the $(n+2)$d term will be $(n-n)$, or 0, and this term will disappear. But if n is negative or fractional, then *no one* of the factors, $(n-1)$, $(n-2)$, $(n-3)$, etc., can be zero, and the expansion may be continued indefinitely. Hence,

1. *When* n *is a positive integer, the expansion of the binomial will be a finite series, the number of terms being* n + 1.

2. *When* n *is negative or fractional, the expansion of the binomial will be an infinite series.*

APPLICATION OF THE BINOMIAL FORMULA.

376. Let us resume the equation,

$$(a+x)^n = a^n + na^{n-1}x + \frac{n(n-1)}{2}a^{n-2}x^2 + \frac{n(n-1)(n-2)}{2 \cdot 3}a^{n-3}x^3 + \dots \quad . \quad . \quad . \quad (c).$$

If n be entire and positive, this formula will be an expression of *involution*, denoting some power of the binomial.

If n be fractional and positive, the formula will be an expression of *evolution*, denoting some root of the binomial.

If n be negative, the formula will express the *reciprocal* of some power or root of the binomial.

377. Since the binomial coefficients depend entirely upon the exponent n, they may be formed independently. To do this, we have simply to commence with unity and multiply by n, $\frac{n-1}{2}$, $\frac{n-2}{3}$, etc., continually.

1. Expand $(a-x)^6$ into a series.

Here $n = 6$; hence,

The first	coefficient	is	1	$= 1$;
" second	"	"	1×6	$= 6$;
" third	"	"	$6 \times \frac{5}{2}$	$= 15$;
" fourth	"	"	$15 \times \frac{4}{3}$	$= 20$;
" fifth	"	"	$20 \times \frac{3}{4}$	$= 15$;
" sixth	"	"	$15 \times \frac{2}{5}$	$= 6$;
" seventh	"	"	$6 \times \frac{1}{6}$	$= 1$.

Since the odd powers of $-x$ are negative, we have for the literal factors of the terms,

$$a^6, \quad -a^5x, \quad +a^4x^2, \quad -a^3x^3, \quad +a^2x^4, \quad -ax^5, \quad +x^6.$$

Therefore the expansion will be

$$(a-x)^6 = a^6 - 6a^5x + 15a^4x^2 - 20a^3x^3 + 15a^2x^4 - 6ax^5 + x^6.$$

2. Expand $(a+x)^{\frac{1}{2}}$ into a series.

In this example $n = \frac{1}{2}$. Represent the coefficients by A, B, C, D, ...; then

$$A = \qquad\qquad +1;$$

$$B = A \times n \qquad = +\frac{1}{2};$$

$$C = B \times \left(\frac{n-1}{2}\right) = -\frac{1}{2}\cdot\frac{1}{4};$$

$$D = C \times \left(\frac{n-2}{3}\right) = +\frac{1}{2}\cdot\frac{1}{4}\cdot\frac{3}{6};$$

$$E = D \times \left(\frac{n-3}{4}\right) = -\frac{1}{2}\cdot\frac{1}{4}\cdot\frac{3}{6}\cdot\frac{5}{8}; \text{ etc.}$$

The literal factors of the terms will be

$$a^{\frac{1}{2}}, \quad a^{-\frac{1}{2}}x, \quad a^{-\frac{3}{2}}x^2, \quad a^{-\frac{5}{2}}x^3, \quad a^{-\frac{7}{2}}x^4, \quad a^{-\frac{9}{2}}x^5, \ldots.$$

Hence, $(a+x)^{\frac{1}{2}} =$

$$a^{\frac{1}{2}} + \tfrac{1}{2}a^{-\frac{1}{2}}x - \frac{1}{2\cdot4}a^{-\frac{3}{2}}x^2 + \frac{3}{2\cdot4\cdot6}a^{-\frac{5}{2}}x^3 - \frac{3\cdot5}{2\cdot4\cdot6\cdot8}a^{-\frac{7}{2}}x^4 + \ldots.$$

or by taking out the factor $a^{\frac{1}{2}}$, in the second member,

$(a+x)^{\frac{1}{2}} =$

$$a^{\frac{1}{2}}\left(1 + \tfrac{1}{2}a^{-1}x - \frac{1}{2\cdot4}a^{-2}x^2 + \frac{3}{2\cdot4\cdot6}a^{-3}x^3 - \frac{3\cdot5}{2\cdot4\cdot6\cdot8}a^{-4}x^4 + \ldots.\right);$$

or by clearing of negative exponents,

$$(a+x)^{\frac{1}{2}} = a^{\frac{1}{2}}\left(1 + \frac{x}{2a} - \frac{x^2}{2\cdot4a^2} + \frac{3x^3}{2\cdot4\cdot6a^3} - \frac{3\cdot5x^4}{2\cdot4\cdot6\cdot8a^4} + \ldots.\right).$$

We might have obtained this last result directly, by putting the binomial in the form of $a^{\frac{1}{2}}\left(1+\frac{x}{a}\right)^{\frac{1}{2}}$. It is well, however, to note the transformations made above.

3. Expand $\frac{1}{(a+x)^2}$ into a series.

Observe that

$$\frac{1}{(a+x)^2} = (a+x)^{-2} = a^{-2}\left(1+\frac{x}{a}\right)^{-2} = \frac{1}{a^2}\left(1+\frac{x}{a}\right)^{-2}.$$

Whence, by expanding the factor $\left(1+\frac{x}{a}\right)^{-2}$, we obtain

$$\frac{1}{(a+x)^2} = \frac{1}{a^2}\left(1 - \frac{2x}{a} + \frac{3x^2}{a^2} - \frac{4x^3}{a^3} + \frac{5x^4}{a^4} - \dots\right).$$

4. Expand $(a^3 - x^2)^5$ into a series.

If we take the descending powers of a^3, commencing with the 5th, and the ascending powers of x^2, commencing with the first, we have for the literal factors of the terms,

$$a^{15},\ a^{12}x^2,\ a^9x^4,\ a^6x^6,\ a^3x^8,\ x^{10}.$$

Hence, with the coefficients the development becomes

$$(a^3 - x^2)^5 = a^{15} - 5a^{12}x^2 + 10a^9x^4 - 10a^6x^6 + 5a^3x^8 - x^{10}.$$

EXAMPLES FOR PRACTICE.

1. Find the fifth power of $a - b$.

Ans. $a^5 - 5a^4b + 10a^3b^2 - 10a^2b^3 + 5ab^4 - b^5$.

2. Find the sixth power of $1 + c$.

Ans. $1 + 6c + 15c^2 + 20c^3 + 15c^4 + 6c^5 + c^6$.

3. Find the seventh power of $x + y$.

Ans. $x^7 + 7x^6y + 21x^5y^2 + 35x^4y^3 + 35x^3y^4 + 21x^2y^5 + 7xy^6 + y^7$.

4. Find the eighth power of $a^2 - 1$.

Ans. $a^{16} - 8a^{14} + 28a^{12} - 56a^{10} + 70a^8 - 56a^6 + 28a^4 - 8a^2 + 1$.

5. Find the ninth power of $a - c$.

Ans. $a^9 - 9a^8c + 36a^7c^2 - 84a^6c^3 + 126a^5c^4 - 126a^4c^5 + 84a^3c^6 - 36a^2c^7 + 9ac^8 - c^9$.

6. Expand $(1 + ax)^5$.

Ans. $1 + 5ax + 10a^2x^2 + 10a^3x^3 + 5a^4x^4 + a^5x^5$.

7. Expand $(a^2 - x^2)^6$.

Ans. $a^{12} - 6a^{10}x^2 + 15a^8x^4 - 20a^6x^6 + 15a^4x^8 - 6a^2x^{10} + x^{12}$.

8. Expand $(x^2 - z^4)^5$.

Ans. $x^{10} - 5x^8z^4 + 10x^6z^8 - 10x^4z^{12} + 5x^2z^{16} - z^{20}$.

9. Expand $(a^2x + dy^2)^6$.

Ans. $a^{12}x^6 + 6a^{10}x^5dy^2 + 15a^8x^4d^2y^4 + 20a^6x^3d^3y^6 + 15a^4x^2d^4y^8 + 6a^2xd^5y^{10} + d^6y^{12}$.

10. Expand $(a - x)^{\frac{1}{2}}$ into a series.

Ans. $\sqrt{a}\left(1 - \frac{x}{2a} - \frac{x^2}{2 \cdot 4a^2} - \frac{3x^3}{2 \cdot 4 \cdot 6a^3} - \frac{3 \cdot 5x^4}{2 \cdot 4 \cdot 6 \cdot 8a^4} - \dots\right)$.

11. Expand $(1 - x)^{\frac{1}{3}}$ into a series.

Ans. $1 - \frac{x}{3} - \frac{2x^2}{3 \cdot 6} - \frac{2 \cdot 5x^3}{3 \cdot 6 \cdot 9} - \frac{2 \cdot 5 \cdot 8x^4}{3 \cdot 6 \cdot 9 \cdot 12} - \frac{2 \cdot 5 \cdot 8 \cdot 11x^5}{3 \cdot 6 \cdot 9 \cdot 12 \cdot 15} - \dots$

12. Expand $(a+1)^{\frac{1}{4}}$ into a series.

Ans. $a^{\frac{1}{4}}\left(1 + \frac{1}{4a} - \frac{3}{4 \cdot 8a^2} + \frac{3 \cdot 7}{4 \cdot 8 \cdot 12a^3} - \frac{3 \cdot 7 \cdot 11}{4 \cdot 8 \cdot 12 \cdot 16a^4} + \dots\right)$.

13. Expand $(a + b)^{\frac{1}{3}}$ into a series.

Ans. $a^{\frac{1}{3}}\left(1 + \frac{b}{3a} - \frac{2b^2}{3 \cdot 6a^2} + \frac{2 \cdot 5b^3}{3 \cdot 6 \cdot 9a^3} - \frac{2 \cdot 5\, 8b^4}{3 \cdot 6 \cdot 9 \cdot 12a^4} + \dots\right)$.

14. Expand $\frac{1}{a - b}$ into a series.

Ans. $\frac{1}{a} + \frac{b}{a^2} + \frac{b^2}{a^3} + \frac{b^3}{a^4} + \frac{b^4}{a^5} + \dots$

15. Expand $\frac{a}{(1 - x)^2}$ into a series.

Ans. $a(1 + 2x + 3x^2 + 4x^3 + 5x^4 + 6x^5 + \dots)$.

16. Expand $(a^2 + b^2)^{\frac{1}{2}}$ into a series.

Ans. $a + \frac{b^2}{2a} - \frac{b^4}{2 \cdot 4a^3} + \frac{3b^6}{2 \cdot 4 \cdot 6a^5} - \frac{3 \cdot 5b^8}{2 \cdot 4 \cdot 6 \cdot 8a^7} + \dots$

17. Expand $(a - c^2)^{\frac{2}{3}}$ into a series.

Ans. $a^{\frac{2}{3}}\left(1 - \frac{2c^2}{3a} - \frac{2c^4}{3 \cdot 6a^2} - \frac{2 \cdot 4c^6}{3 \cdot 6 \cdot 9a^3} - \frac{2 \cdot 4 \cdot 7c^8}{3 \cdot 6 \cdot 9 \cdot 12a^4} - \dots\right)$.

18. Expand $d(c^2 + x^2)^{-\frac{1}{2}}$ into a series.

Ans. $\frac{d}{c}\left(1 - \frac{x^2}{2c^2} + \frac{3x^4}{2 \cdot 4c^4} - \frac{3 \cdot 5x^6}{2 \cdot 4 \cdot 6c^6} + \frac{3 \cdot 5 \cdot 7x^8}{2 \cdot 4 \cdot 6 \cdot 8c^8} - \dots\right)$.

19. Expand $(1-a)^{-3}$ into a series.

Ans. $1+3a+6a^2+10a^3+15a^4+21a^5+28a^6+36a^7+\ldots.$

20. Expand $(a^2-x^2)^{\frac{1}{4}}$ into a series.

$$Ans.\ \sqrt{a}\left(a-\frac{3x^2}{4a}-\frac{3x^4}{4\cdot 8a^3}-\frac{3\cdot 5x^6}{4\cdot 8\cdot 12a^5}-\frac{3\cdot 5\cdot 9x^8}{4\cdot 8\cdot 12\cdot 16a^7}-\ldots\right).$$

21. Expand $(a+y)^{-4}$ into a series.

$$Ans.\ \frac{1}{a^4}-\frac{4y}{a^5}+\frac{10y^2}{a^6}-\frac{20y^3}{a^7}+\frac{35y^4}{a^8}-\frac{56y^5}{a^9}+\ldots.$$

22. Expand $\dfrac{r}{\sqrt[5]{1-r}}$ into a series.

$$Ans.\ r+\frac{r^2}{5}+\frac{6r^3}{2\cdot 5^2}+\frac{6\cdot 11r^4}{2\cdot 3\cdot 5^3}+\frac{6\cdot 11\cdot 16r^5}{2\cdot 3\cdot 4\cdot 5^4}+\ldots.$$

23. Expand $\sqrt[15]{1-x^4}$ into a series.

$$Ans.\ 1-\frac{x^4}{15}-\frac{14x^8}{2\cdot 15^2}-\frac{14\cdot 29x^{12}}{2\cdot 3\cdot 15^3}-\frac{14\cdot 29\cdot 44x^{16}}{2\cdot 3\cdot 4\cdot 15^4}-\ldots.$$

METHOD OF SUBSTITUTION.

378. In the formula $(x+y)^n =$

$$x^n+nx^{n-1}y+\frac{n(n-1)}{2}x^{n-2}y^2+\frac{n(n-1)(n-2)}{2\cdot 3}x^{n-3}y^3+\ldots.$$

we may suppose x and y to represent any quantities whatever; and thus we may obtain the development of the powers of binomials with numerical coefficients, or of polynomials.

1. Involve $3a+2c$ to the fifth power.

The binomial coefficients for the fifth power are

$$1,\quad 5,\quad 10,\quad 10,\quad 5,\quad 1.$$

And by connecting these with the powers of the given terms, according to the law of the formula, we have

$(3a+2c)^5=(3a)^5+5(3a)^4(2c)+10(3a)^3(2c)^2+10(3a)^2(2c)^3+5(3a)(2c)^4+(2c)^5;$

or, by performing the operations indicate

$$(3a+2c)^5=243a^5+810a^4c+1080a^3c^2+720a^2c^3+240ac^4+32c^5.$$

2. Involve $a + b + 2c^2$ to the fourth power.

We may consider the polynomial in two parts, $a + b$, represented by x, and $+ 2c^2$ represented by y. Then we have

$(a + b + 2c^2)^4 = (a + b)^4 + 4(a+b)^3(2c^2) + 6(a + b)^2(2c^2)^2 + 4(a + b)(2c^2)^3 + (2c^2)^4$.

Performing the operations indicated,

$(a + b + 2c^2)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4 + 8a^3c^2 + 24a^2bc^2 + 24ab^2c^2 + 8b^3c^2 + 24a^2c^4 + 48abc^4 + 24b^2c^4 + 32ac^6 + 32bc^6 + 16c^8$.

EXAMPLES FOR PRACTICE.

1. Find the third power of $a - 2b$.

Ans. $a^3 - 6a^2b + 12ab^2 - 8b^3$.

2. Find the fourth power of $2a + 3x$.

Ans. $16a^4 + 96a^3x + 216a^2x^2 + 216ax^3 + 81x^4$.

3. Find the fourth power of $1 - \frac{1}{2}a$.

Ans. $1 - 2a + \frac{3}{2}a^2 - \frac{1}{2}a^3 + \frac{1}{16}a^4$.

4. Find the fourth power of $a^2 - ax + x^2$.

Ans. $a^8 - 4a^7x + 10a^6x^2 - 16a^5x^3 + 19a^4x^4 - 16a^3x^5 + 10a^2x^6 - 4ax^7 + x^8$.

5. Expand $(4a^2 - 3x)^{\frac{1}{2}}$ into a series.

$$Ans.\ \sqrt{2a}\left(1 - \frac{3x}{16a^2} - \frac{27x^2}{512a^4} - \frac{567x^3}{24576a^6} - \ldots\right).$$

379. When a binomial having numerical coefficients is to be raised to any power, the coefficients of the expansion may be obtained with great facility by means of a simple modification of the binomial formula. We have $(z + u)^n =$

$$z^n + nz^{n-1}u + \frac{n(n-1)}{2}z^{n-2}u^2 + \frac{n(n-1)(n-2)}{2 \cdot 3}z^{n-3}u^3 + \ldots.$$

In this equation make $z = ax$, and $u = by$; then

$$(ax + by)^n = a^nx^n + na^{n-1}b \cdot x^{n-1}y + \frac{n(n-1)}{2}a^{n-2}b^2 \cdot x^{n-2}y^2 + \frac{n(n-1)(n-2)}{2 \cdot 3}a^{n-3}b^3 \cdot x^{n-3}y^3 + \ldots,$$

in which a and b may represent the numerical coefficients of x and y. Now denote the numerical coefficients of the expansion by C_1, C_2, C_3, etc. We shall then have

$$(ax + by)^n = C_1x^n + C_2x^{n-1}y + C_3x^{n-2}y^2 + C_4x^{n-3}y^3 + \dots,$$

in which

$$\begin{aligned}
C_1 &= a^n, \\
C_2 &= C_1 \cdot \frac{n}{1} \cdot \frac{b}{a}, \\
C_3 &= C_2 \cdot \frac{n-1}{2} \cdot \frac{b}{a}, \\
C_4 &= C_3 \cdot \frac{n-2}{3} \cdot \frac{b}{a}, \\
C_5 &= C_4 \cdot \frac{n-3}{4} \cdot \frac{b}{a}.
\end{aligned}$$

1. Find the fourth power of $5a + 3x$.

In this example we have

$$n = 4, \quad a = 5, \quad b = 3,$$

and the coefficients of the expansion will be

$$\begin{aligned}
C_1 &= 5^4 &&= 625, \\
C_2 &= 625 \cdot \tfrac{4}{1} \cdot \tfrac{3}{5} &&= 1500, \\
C_3 &= 1500 \cdot \tfrac{3}{2} \cdot \tfrac{3}{5} &&= 1350, \\
C_4 &= 1350 \cdot \tfrac{2}{3} \cdot \tfrac{3}{5} &&= 540, \\
C_5 &= 540 \cdot \tfrac{1}{4} \cdot \tfrac{3}{5} &&= 81.
\end{aligned}$$

Hence,

$$(5a + 3x)^4 = 625a^4 + 1500a^3x + 1350a^2x^2 + 540ax^3 + 81x^4, \ \textit{Ans.}$$

2. Find the fourth power of $\dfrac{2c}{3} - \dfrac{4x}{5}$.

We have $n = 4$, $a = \dfrac{2}{3}$, $b = \dfrac{4}{5}$, and $\dfrac{b}{a} = \dfrac{4}{5} \cdot \dfrac{3}{2} = \dfrac{6}{5}$.

Hence the coefficients of the expansion are

$$\begin{aligned}
C_1 & \quad (\tfrac{2}{3})^4 &&= \tfrac{16}{81}, \\
C_2 &= \tfrac{16}{81} \cdot \tfrac{4}{1} \cdot \tfrac{6}{5} &&= \tfrac{128}{135}, \\
C_3 &= \tfrac{128}{135} \cdot \tfrac{3}{2} \cdot \tfrac{6}{5} &&= \tfrac{128}{75}, \\
C_4 &= \tfrac{128}{75} \cdot \tfrac{2}{3} \cdot \tfrac{6}{5} &&= \tfrac{512}{375}, \\
C_5 &= \tfrac{512}{375} \cdot \tfrac{1}{4} \cdot \tfrac{6}{5} &&= \tfrac{256}{625}.
\end{aligned}$$

Hence,

$$\left(\frac{2c}{3} - \frac{4x}{5}\right)^4 = \frac{16}{81}c^4 - \frac{128}{135}c^3x + \frac{128}{75}c^2x^2 - \frac{512}{375}cx^3 + \frac{256}{625}x^4.$$

EXAMPLES FOR PRACTICE.

1. Find the fourth power of $2x + 5y$.

Ans. $16x^4 + 160x^3y + 600x^2y^2 + 1000xy^3 + 625y^4$.

2. Find the fifth power of $2a - 3x$.

Ans. $32a^5 - 240a^4x + 720a^3x^2 - 1080a^2x^3 + 810ax^4 - 243x^5$.

3. Find the sixth power of $3 + 4x^2$.

Ans. $729 + 5832x^2 + 19440x^4 + 34560x^6 + 34560x^8 + 18432x^{10} + 4096x^{12}$.

Find the fourth power of $\frac{3a}{4} + \frac{4r}{5}$.

Ans. $\frac{81}{256}a^4 + \frac{27}{20}a^3r + \frac{54}{25}a^2r^2 + \frac{192}{125}ar^3 + \frac{256}{625}r^4$.

5. Find the sixth power of $\frac{2t}{3} + \frac{3r}{2}$.

Ans. $\frac{64}{729}t^6 + \frac{32}{27}t^5r + \frac{20}{3}t^4r^2 + 20t^3r^3 + \frac{135}{4}t^2r^4 + \frac{243}{8}tr^5 + \frac{729}{64}r^6$.

6. Find the fifth power of $\frac{m}{4} - \frac{1}{5}$.

$$Ans.\ \frac{m^5}{1024} - \frac{m^4}{256} + \frac{m^3}{160} - \frac{m^2}{200} + \frac{m}{500} - \frac{1}{3125}.$$

7. Find the eighth power of $\frac{m}{2} - \frac{1}{2m}$.

$$Ans.\ \frac{m^8}{256} - \frac{m^6}{32} + \frac{7m^4}{64} - \frac{7m^2}{32} + \frac{35}{128} - \frac{7}{32m^2} + \frac{7}{64m^4} - \frac{1}{32m^6} + \frac{1}{256m^8}.$$

DEVELOPMENT OF SURD ROOTS INTO SERIES.

380. The approximate value of a surd root may be obtained with much facility by expanding the root into a series.

Let a^n represent that perfect n^{th} power, which is next less or next greater than the given number, and let b denote the difference between this power and the given number. Then

$$a^n + b, \quad \text{or} \quad a^n - b,$$

will express the given number. But we have

$$\sqrt[n]{a^n + b} = a\sqrt[n]{1 + \frac{b}{a^n}}, \quad \text{and} \quad \sqrt[n]{a^n - b} = a\sqrt[n]{1 - \frac{b}{a^n}}.$$

Developing the radical parts into series, we have

$$\sqrt[n]{a^n + b} = a\left(1 + \frac{1}{n}\cdot\frac{b}{a^n} + \frac{1}{n}\cdot\frac{1-n}{2n}\cdot\frac{b^2}{a^{2n}} + \frac{1}{n}\cdot\frac{1-n}{2n}\cdot\frac{1-2n}{3n}\cdot\frac{b^3}{a^{3n}} + \ldots\right) \quad . \; . \; . \; (1).$$

$$\sqrt[n]{a^n - b} = a\left(1 - \frac{1}{n}\cdot\frac{b}{a^n} + \frac{1}{n}\cdot\frac{1-n}{2n}\cdot\frac{b^2}{a^{2n}} - \frac{1}{n}\cdot\frac{1-n}{2n}\cdot\frac{1-2n}{3n}\cdot\frac{b^3}{a^{3n}} + \ldots\right) \quad . \; . \; . \; (2).$$

The second members of these equations contain no radicals; hence,

Any surd may be developed into a series of rational terms; whence by summing the series, we may obtain approximately the indicated root.

It should be observed that the smaller the fraction $\frac{b}{a^n}$ is, the more rapidly will the series converge.

1. Find the cube root of 76, to six decimal places.

The smallest fraction will result by taking the cube which is next *less* than 76, or 64; thus,

$$\sqrt[3]{76} = \sqrt[3]{64 + 12} = 4\sqrt[3]{1 + \frac{12}{64}} = 4\sqrt[3]{1 + \frac{3}{16}}.$$

We may now develop the radical part by equation (1), in which

$$n = 3, \quad a = 4, \quad \frac{b}{a^n} = \frac{3}{16}.$$

To form the binomial coefficients we have the factors,

$$\frac{1}{n} = \frac{1}{3},$$

$$\frac{1-n}{2n} = -\frac{1}{3}, \qquad \frac{1-4n}{5n} = -\frac{11}{15},$$

$$\frac{1-2n}{3n} = -\frac{5}{9}, \qquad \frac{1-5n}{6n} = -\frac{7}{9},$$

$$\frac{1-3n}{4n} = -\frac{2}{3}, \qquad \frac{1-6n}{7n} = -\frac{17}{21}, \text{ etc.}$$

We represent the successive terms by A, B, C, etc.; and to secure accuracy in the final result to the 6th place of decimals,

we should carry the computation in each term to the 7th place. Thus we have

$$
\begin{array}{lcl}
A & & = +1.0000000 \\
B = +\frac{1}{3}\cdot\frac{3}{16} & & = +\ .0625000 \\
C = -\frac{1}{3}\cdot\frac{3}{16}\cdot B & & = -\ .0039062 \\
D = -\frac{5}{9}\cdot\frac{3}{16}\cdot C & & = +\ .0004069 \\
E = -\frac{2}{3}\cdot\frac{3}{16}\cdot D & & = -\ .0000508 \\
F = -\frac{11}{15}\cdot\frac{3}{16}\cdot E & & = +\ .0000069 \\
G = -\frac{7}{9}\cdot\frac{3}{16}\cdot F & & = -\ .0000010 \\
H = -\frac{17}{21}\cdot\frac{3}{16}\cdot G & & = +\ \underline{.0000001} \\
\text{Algebraic sum,} & & \quad 1.0589559 = \sqrt{1+\frac{3}{16}}. \\
 & & \quad\quad\quad\quad\ \ \underline{4}
\end{array}
$$

Whence, $\sqrt{76} = 4.235824 \pm$, *Ans.*

2. Find the 5th root of 25, to 6 decimal places.

The most convenient fraction will result by taking that 5th power which is next *greater* than 25, or 32; thus,

$$\sqrt[5]{25} = \sqrt[5]{32-7} = 2\sqrt[5]{1-\frac{7}{32}}.$$

Equation (2) will now apply; and the operation will be as follows:

$$\frac{1}{n} = \frac{1}{5},$$

$$\frac{1-n}{2n} = -\frac{2}{5}, \qquad \frac{1-4n}{5n} = -\frac{19}{25},$$

$$\frac{1-2n}{3n} = -\frac{3}{5}, \qquad \frac{1-5n}{6n} = -\frac{4}{5},$$

$$\frac{1-3n}{4n} = -\frac{7}{10}, \qquad \frac{1-6n}{7n} = -\frac{29}{35}, \text{ etc.}$$

$$
\begin{array}{lcl}
A & & = +1.0000000 \\
B = -\frac{1}{5}\cdot\frac{7}{32} & & = -\quad 437500 \\
C = +\frac{2}{5}\cdot\frac{7}{32}\cdot B & & = -\quad\ 38281 \\
D = +\frac{3}{5}\cdot\frac{7}{32}\cdot C & & = -\quad\quad 5024 \\
E = +\frac{7}{10}\cdot\frac{7}{32}\cdot D & & = -\quad\quad\ 769 \\
F = +\frac{19}{25}\cdot\frac{7}{32}\cdot E & & = -\quad\quad\ 128 \\
G = +\frac{4}{5}\cdot\frac{7}{32}\cdot F & & = -\quad\quad\quad 22 \\
H = +\frac{29}{35}\cdot\frac{7}{32}\cdot G & & = -\quad\quad\quad \underline{4} \\
\text{Algebraic sum,} & & \quad .9518272 = \sqrt[5]{1-\frac{7}{32}}. \\
 & & \quad\quad\quad\quad\ \ \underline{2}
\end{array}
$$

Whence, $\sqrt[5]{25} = 1.903654+$, *Ans.*

EXAMPLES FOR PRACTICE.

Find the values of the following indicated roots, to the 6th decimal place:

1. $\sqrt[3]{9}$. *Ans.* 2.080084.
2. $\sqrt[3]{31}$. *Ans.* 3.141381.
3. $\sqrt[3]{100}$. *Ans.* 4.641589.
4. $\sqrt[3]{110}$. *Ans.* 4.791420.
5. $\sqrt[5]{297}$. *Ans.* 3.122851.
6. $\sqrt[6]{60}$. *Ans.* 1.978602.
7. $\sqrt[5]{4}$. *Ans.* 1.319508.
8. $\sqrt[5]{3275}$. *Ans.* 5.047104.
9. $\sqrt[7]{125}$. *Ans.* 1.993235.

EXPANSION OF FRACTIONS INTO SERIES.

381. An irreducible fraction may always be converted into a series, by dividing the numerator by the denominator.

convert $\dfrac{1}{1+a}$ into a series.

Observe that $\dfrac{1}{1+a} = \dfrac{1}{a+1}$.

Hence, there may be two ways of dividing:

1st. 2d.

$$\begin{array}{l}
1+a)\,1 \quad (1-a+a^2-\ldots. \\
\underline{1+a} \\
\quad -a \\
\quad \underline{-a-a^2} \\
\qquad +a^2
\end{array}
\qquad\qquad
\begin{array}{l}
a+1)\,1 \quad \left(\frac{1}{a}-\frac{1}{a^2}+\frac{1}{a^3}-\ldots.\right. \\
\underline{1+\frac{1}{a}} \\
\quad -\frac{1}{a} \\
\quad \underline{-\frac{1}{a}-\frac{1}{a^2}} \\
\qquad +\frac{1}{a^2}
\end{array}$$

The law of expansion is obvious in both quotients, and we have from the same fraction two series; thus,

$$\frac{1}{1+a} = 1 - a + a^2 - a^3 + a^4 - a^5 + \ldots \quad (1),$$

$$\frac{1}{a+1} = \frac{1}{a} - \frac{1}{a^2} + \frac{1}{a^3} - \frac{1}{a^4} + \frac{1}{a^5} - \frac{1}{a^6} + \ldots \quad (2).$$

We observe that each of the series obtained is by its law of development an infinite series.

EXAMPLES FOR PRACTICE.

1. Convert $\frac{a}{a+x}$ into an infinite series.

   Ans. $1 - \frac{x}{a} + \frac{x^2}{a^2} - \frac{x^3}{a^3} + \frac{x^4}{a^4} - \ldots$

2. Convert $\frac{a}{a-x}$ into an infinite series.

   Ans. $1 + \frac{x}{a} + \frac{x^2}{a^2} + \frac{x^3}{a^3} + \frac{x^4}{a^4} + \ldots$

3. Convert $\frac{1+x}{1-x}$ into an infinite series.

   Ans. $1 + 2x + 2x^2 + 2x^3 + 2x^4 + \ldots$

4. Convert $\frac{a+x}{a^2+x^2}$ into an infinite series.

   Ans. $\frac{1}{a} - \frac{x^2}{a^3} + \frac{x^4}{a^5} - \frac{x^6}{a^7} + \frac{x^8}{a^9} - \ldots$

5. Convert $\frac{1}{1-a+a^2}$ into an infinite series.

   Ans. $1 + a - a^3 - a^4 + a^6 + a^7 - a^9 - a^{10} + \ldots$

6. Convert $\frac{1-x}{1-2x-3x^2}$ into an infinite series.

   Ans. $1 + x + 5x^2 + 13x^3 + 41x^4 + \ldots$

7. Convert $\frac{1+2x}{1-x-x^2}$ into an infinite series.

   Ans. $1 + 3x + 4x^2 + 7x^3 + 11x^4 + 18x^5 + \ldots$

METHOD OF INDETERMINATE COEFFICIENTS.

382. It is evident that if a fraction be developed into a series, the equation which results by placing the fraction equal to its development is capable of being satisfied for any value of the unknown quantity ; in other words, *it is an identical equation.*

On this fact depends an important method of expanding an algebraic expression into a series, called the *Method of Indeterminate Coefficients.* It consists in *assuming* the required development in the form of a series with unknown coefficients, and afterward determining the values of the coefficients by means of the properties of identical equations.

1. Develop $\dfrac{1+2x}{1-3x}$ into an infinite series.

Assume

$$\frac{1+2x}{1-3x}=A+Bx+Cx^2+Dx^3+Ex^4+\dots \quad . \quad . \quad . \quad (1).$$

Clearing this equation of fractions, and transposing all the terms to the second member,

$$0=(A-1)+\left.\begin{array}{c}B\\-3A\\-2\end{array}\right|x+\left.\begin{array}{c}C\\-3B\end{array}\right|x^2+\left.\begin{array}{c}D\\-3C\end{array}\right|x^3+\left.\begin{array}{c}E\\-3D\end{array}\right|x^4+\dots \quad . \quad . \quad . \quad (2).$$

The term $A-1$ may here be considered as the coefficient of x^0 understood.

Now because (2) is an identical equation, the coefficients of the different powers of x are separately equal to zero (**368**, IV). Thus,

$$\begin{array}{rlcrl}
A-1 & =0, & \text{whence} & A= & 1\,; \\
B-3A-2 & =0, & \text{"} & B= & 5\,; \\
C-3B & =0, & \text{"} & C= & 15\,; \\
D-3C & =0, & \text{"} & D= & 45\,; \\
E-3D & =0, & \text{"} & E= & 135, \text{ etc.}
\end{array}$$

Substituting these values in (1),

$$\frac{1+2x}{1-3x}=1+5x+15x^2+45x^3+135x^4+\dots$$

2. Develop $\dfrac{1+x}{x-2x^2+6x^3}$ into an infinite series.

We perceive that the first term of the series must be $\dfrac{1}{x}$ or x^{-1}. Therefore, assume

$$\frac{1+x}{x-2x^2+6x^3} = Ax^{-1} + Bx^0 + Cx + Dx^2 + Ex^3 + Fx^4 + \ldots.$$

Clearing of fractions and transposing,

$$0 = \begin{array}{r|r|r|r|r|r|l} A & x^0 + \;B & x + \;C & x^2 + \;D & x^3 + \;E & x^4 + \;F & x^5 \ldots. \\ -1 & -2A & -2B & -2C & -2D & -2E & \\ & -1 & +6A & +6B & +6C & +6D & \end{array}$$

Putting the coefficients equal to zero,

$$\begin{array}{rlrl} A - 1 &= 0, & \text{whence,}\quad A = & 1; \\ B - 2A - 1 &= 0, & \text{"}\quad B = & 3; \\ C - 2B + 6A &= 0, & \text{"}\quad C = & 0; \\ D - 2C + 6B &= 0, & \text{"}\quad D = & -\ 18; \\ E - 2D + 6C &= 0, & \text{"}\quad E = & -\ 36; \\ F - 2E + 6D &= 0, & \text{"}\quad F = & +\ 36; \\ G - 2F + 6E &= 0, & \text{"}\quad G = & +\ 288, \text{ etc.} \end{array}$$

Substituting these values in the assumed development, and observing that the term containing C will disappear because $C = 0$, we have

$$\frac{1+x}{x-2x^2+6x^3} = \frac{1}{x} + 3 - 18x^2 - 36x^3 + 36x^4 + 288x^5 \ldots.$$

Note.—It is not necessary to transpose the terms to one member; for if neither member is zero, we have simply to equate the coefficients of the like powers of x in the two members, according to the third property of identical equations.

The method of Indeterminate Coefficients is applicable to a great variety of examples, but always with this provision, viz.: *That we determine by inspection what power of the variable will be contained in the first term of the expansion, and make the first term of the assumed development correspond to the known fact.*

If the assumed development commence with a power of the variable *higher* than it should, the fact will be indicated by an absurdity in one of the resulting equations. If, however, the assumed development commence with a power of the variable

lower than is necessary, no absurdity will arise; but the redundant terms will disappear by reason of the coefficients reducing to zero.

EXAMPLES FOR PRACTICE.

1. Develop $\frac{1-2x}{1-3x}$ into a series.

$$Ans.\ 1 + x + 3x^2 + 9x^3 + 27x^4 + 81x^5 + \ldots.$$

2. Develop $\frac{1+2x}{1-x-x^2}$ into a series.

$$Ans.\ 1 + 3x + 4x^2 + 7x^3 + 11x^4 + 18x^5 + \ldots.$$

3. Develop $\frac{1-x}{1-3x-2x^2}$ into a series.

$$Ans.\ 1 + 2x + 8x^2 + 28x^3 + 100x^4 + 356x^5 + \ldots.$$

4. Develop $\frac{x(1+5x)}{(1-2x)^2}$ into a series.

$$Ans.\ x + 9x^2 + 32x^3 + 92x^4 + 240x^5 + \ldots.$$

5. Develop $\frac{2}{3x-2x^2}$ into a series.

$$Ans.\ \frac{2}{3x} + \frac{4}{9} + \frac{8x}{27} + \frac{16x^2}{81} + \frac{32x^3}{243} + \ldots.$$

6. Develop $\frac{1}{1+2x^2+3x^4}$ into a series.

$$Ans.\ 1 - 2x^2 + x^4 + 4x^6 - 11x^8 + 10x^{10} + 13x^{12} - \ldots.$$

7. Develop $\frac{1+x}{(1+ax)^2}$ into a series.

$$Ans.\ 1+(1-2a)x-(2a-3a^2)x^2+(3a^2-4a^3)x^3- \ldots.$$

8. Develop $\sqrt{1-x}$ into a series.

$$Ans.\ 1 - \frac{x}{2} - \frac{x^2}{2\cdot4} - \frac{3x^3}{2\cdot4\cdot6} - \frac{3\cdot5x^4}{2\cdot4\cdot6\cdot8} - \ldots.$$

NOTE.—Assume $\sqrt{1-x} = A + Bx + Cx^2 + Dx^3 + \ldots.$; then square both members, and the equations for the coefficients will be readily obtained.

9. Develop $\sqrt{1 + 3x + 5x^2 + 7x^3 + 9x^4 + \ldots}$ into a series of rational terms.

$$Ans.\ 1 + \frac{3x}{2} + \frac{11x^2}{8} + \frac{23x^3}{16} + \frac{179x^4}{128} + \ldots$$

10. Develop $\dfrac{1 - 2x^2 + 3x^4 - 4x^6 + 5x^8 - 6x^{10} + \ldots}{1 + x^2 + x^4 + x^6 + x^8 + x^{10} + \ldots}$ into a series.

$$Ans.\ 1 - 3x^2 + 5x^4 - 7x^6 + 9x^8 - 11x^{10} + \ldots$$

REVERSION OF SERIES.

383. The ***Reversion of a Series*** is the process of finding the value of the unknown quantity in the series, expressed in terms of another unknown quantity.

1. Given $y = ax + bx^2 + cx^3 + dx^4 + ex^5 + \ldots$, to find the value of x in terms containing the ascending powers of y.

In this equation, x and y are two indeterminate quantities, and either may have any value whatever without altering the form of the series. We may therefore apply the method of Indeterminate Coefficients. Assume

$$x = Ay + By^2 + Cy^3 + Dy^4 + Ey^5 + \ldots \quad . \quad . \quad . \quad (1).$$

We may now find by involution the values of x^2, x^3, x^4, x^5, etc., carrying each result only to the term containing y^5. Then substituting for x, x^2, x^3, etc., in the given equation, we shall have, after transposing y,

$$\begin{array}{r|r|r|r|r|l}
0 = aA & y + \quad aB & y^2 + \quad aC & y^3 + \quad aD & y^4 + \quad aE & y^5 \ldots \\
-1 & bA^2 & + 2bAB & + 2bAC & + 2bAD & \\
 & & + \quad cA^3 & + \quad bB^2 & + 2bBC & \\
 & & & + 3cA^2B & + 3cA^2C & \\
 & & & + \quad dA^4 & + 3cAB^2 & \\
 & & & & + 4dA^3B & \\
 & & & & + \quad eA^5 &
\end{array}$$

This is an identical equation, being true for all values of y. And if we place the coefficients of the different powers of y separately equal to zero (**368**, IV), and reduce the resulting

equations, we shall obtain the values of the assumed coefficients as follows:

$$(F)\quad\begin{cases} A = \dfrac{1}{a}, \\ B = -\dfrac{b}{a^3}, \\ C = \dfrac{2b^2 - ac}{a^5}, \\ D = -\dfrac{5b^3 - 5abc + a^2d}{a^7}, \\ E = \dfrac{14b^4 - 21ab^2c + 6a^2bd + 3a^2c^2 - a^3e}{a^9}. \end{cases}$$

If we substitute these values of A, B, C, etc., in (1), we shall have the value of x in terms of a, b, c ...., and the powers of y; that is, the given series will be *reversed.*

2. Given $y = ax + bx^3 + cx^5 + dx^7 + ex^9 + \ldots.$, to find the value of x in terms of y.

Assume $x = Ay + By^3 + Cy^5 + Dy^7 + Ey^9 + \ldots. \quad . \quad . \quad . \quad (1).$

Proceeding as before, we shall obtain

$$(G)\quad\begin{cases} A = \dfrac{1}{a}, \\ B = -\dfrac{b}{a^4}, \\ C = \dfrac{3b^2 - ac}{a^7}, \\ D = -\dfrac{12b^3 - 8abc + a^2d}{a^{10}}, \\ E = \dfrac{55b^4 - 55ab^2c + 10a^2bd + 5a^2c^2 - a^3e}{a^{13}}. \end{cases}$$

In the preceding examples the letters a, b, c, ..., represent any coefficients whatever. Hence, in reverting any series in either of these forms, we may determine the values of the assumed coefficients by an application of formula (F), or (G).

3. Revert the series $y = x + 2x^2 + 4x^3 + 8x^4 + \ldots.$

Assume $x = Ay + By^2 + Cy^3 + Dy^4 + \ldots.$

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If we now substitute in formula (F),

$$a = 1, \quad b = 2, \quad c = 4, \quad d = 8,$$

we shall obtain $\quad A = 1, \quad B = -2, \quad C = 4, \quad D = -8.$

Hence, $\quad x = y - 2y^2 + 4y^3 - 8y^4 + \ldots.,$ *Ans.*

EXAMPLES FOR PRACTICE.

1. Revert the series $y = x + x^2 + x^3 + x^4 + x^5 + \ldots.$

   Ans. $x = y - y^2 + y^3 - y^4 + y^5 - \ldots.$

2. Revert the series $y = x + 3x^2 + 5x^3 + 7x^4 + 9x^5 + \ldots.$

   Ans. $x = y - 3y^2 + 13y^3 - 67y^4 + 381y^5 - \ldots.$

3. Revert the series $x = y - \dfrac{y^2}{2} + \dfrac{y^3}{3} - \dfrac{y^4}{4} + \dfrac{y^5}{5} - \ldots.$

   Ans. $y = x + \dfrac{x^2}{1 \cdot 2} + \dfrac{x^3}{1 \cdot 2 \cdot 3} + \dfrac{x^4}{1 \cdot 2 \cdot 3 \cdot 4} + \dfrac{x^5}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} + \ldots.$

4. Revert the series $y = x - x^3 + x^5 - x^7 + x^9 - x^{11} + \ldots.$

   Ans. $x = y + y^3 + 2y^5 + 5y^7 + 14y^9 + \ldots.$

5. Revert the series $y = 2x + 3x^3 + 4x^5 + 5x^7 + \ldots.$

   Ans. $x = \dfrac{y}{2} - \dfrac{3y^3}{16} + \dfrac{19y^5}{128} - \dfrac{152y^7}{1024} + \ldots.$

6. Revert the series $x = 2y + 4y^2 + 6y^3 + 8y^4 + 10y^5 + \ldots.$

   Ans. $y = \dfrac{x}{2} - \dfrac{x^2}{2} + \dfrac{5x^3}{8} - \dfrac{7x^4}{8} + \dfrac{21x^5}{16} - \ldots.$

384. One of the principal objects in reverting a series is, to obtain the approximate value of the unknown quantity when the sum of the series is known. Thus,

1. Given $\dfrac{1}{4} = 2x - \dfrac{4x^2}{3} + \dfrac{6x^3}{5} - \dfrac{8x^4}{7} + \ldots.$ to find the approximate value of x.

Let us put $\quad s = 2x - \dfrac{4x^2}{3} + \dfrac{6x^3}{5} - \dfrac{8x^4}{7} + \ldots. \quad . \quad . \quad . \quad (1),$

and consider x and s as variables. Reverting (1) by formula (F),

$$x = \frac{s}{2} + \frac{s^2}{6} + \frac{13s^3}{360} + \frac{5s^4}{1512} + \ldots. \quad \ldots \quad . \quad (2).$$

Now if we put $s = \frac{1}{4}$ in this equation, the result will be a *con-*

verging series; and we may find the approximate value of x, by computing the values of the terms separately. Thus,

$$\frac{s}{2} = \frac{1}{4} \cdot \frac{1}{2} = .125000,$$

$$\frac{s^2}{6} = \frac{1}{16} \cdot \frac{1}{6} = .010416,$$

$$\frac{13s^3}{360} = \frac{1}{64} \cdot \frac{13}{360} = .000564,$$

$$\frac{5s^4}{1512} = \frac{1}{256} \cdot \frac{5}{1512} = .000013.$$

Hence, $\qquad x = .135993,$ *Ans.*

EXAMPLES.

1. Given $\frac{2}{3} = 5x - 20x^2 + 80x^3 - 320x^4 + 1280x^5 - \ldots$, to find the approximate value of x. *Ans.* $x = .117647$.

2. Given $\frac{1}{2} = x + \frac{x^2}{2\cdot 3} + \frac{x^3}{3\cdot 4} + \frac{x^4}{4\cdot 5} + \frac{x^5}{5\cdot 6} + \ldots$, to find the approximate value of x. *Ans.* .454620.

3. Given $\frac{1}{5} = x - \frac{x^3}{6} - \frac{x^5}{40} - \frac{x^7}{112} - \ldots$, to find the approximate value of x. *Ans.* $x = .201369$.

4. Given $\frac{1}{4} = x - \frac{3x^2}{2\cdot 4} + \frac{5x^3}{4\cdot 6} - \frac{7x^4}{6\cdot 8} + \ldots$, to find the approximate value of x. *Ans.* $x = .274655$.

SUMMATION OF INFINITE SERIES.

385. The ***Summation of a Series*** is the process of obtaining a finite expression equivalent to the series.

386. The method of summing a given series must always depend upon the nature of the series, or the law governing its development. Formulas have already been given for the summation of arithmetical and geometrical series. We will now investigate the methods of summing a variety of other series.

RECURRING SERIES.

387. A ***Recurring Series*** is one in which there is a fixed relation between any term, or two or more consecutive terms, and the term which immediately succeeds. Thus,

$$1 + 4x + 11x^2 + 34x^3 + 101x^4 + \dots.$$

is a recurring series, in which if any two consecutive terms be taken, the product of the first by $3x^2$ plus the product of the second by $2x$, will be equal to the next succeeding term. The coefficients of these multipliers, or

$$3, \; 2,$$

are called the *scale of relation*. In the recurring series

$$1 + x + 3x^2 + 8x^3 + 17x^4 + 42x^5 + 100x^6 + \dots.$$

the scale of relation is $\quad 3, \; 2, \; 1.$

388. A recurring series is said to be of the *first order*, when each term after the first depends upon the term which immediately precedes it. The scale of relation will consist of a single part, and the series will be a geometrical progression.

A recurring series is said to be of the *second order*, when each term after the second depends upon the two preceding terms; the scale of relation consists of two parts.

A recurring series is said to be of the *third order*, when each term after the third depends upon the three preceding terms; the scale of relation consists of three parts.

389. To find the scale of relation and the sum of a recurring series of the second order.

1st. Let $a, b, c, d,$ represent the coefficients of any four consecutive terms; and let $m, n,$ denote the scale of relation. Then from the nature of the series, we have

$$\left.\begin{aligned} ma + nb &= c \\ mb + nc &= d \end{aligned}\right\} \quad \dots\dots\dots \quad (P).$$

These two equations will determine the values of m and n.

2d. To find the sum of the series, denote the *terms* of the series by $A, B, C,$ etc., and let

$$S = A + B + C + D + E + \dots. \quad \dots \quad (1).$$

The series is supposed to contain the ascending powers of x, the first power occurring in either A or B. Then because the series is of the second order, we have

$$\left.\begin{aligned} C &= mAx^2 + nBx \\ D &= mBx^2 + nCx \\ E &= mCx^2 + nDx \\ \text{etc.} &\quad \text{etc.} \quad \text{etc.} \end{aligned}\right\} \quad \ldots\ldots \quad (2).$$

Adding these equations, and observing the value of S in (1), we have

$$S - A - B = mx^2S + nx(S - A);$$

whence,

$$S = \frac{A + B - Anx}{1 - nx - mx^2} \quad \ldots\ldots \quad (Q).$$

390. To find the scale of relation and the sum of a recurring series of the third order.

1st. Let a, b, c, d, e, f, represent the coefficients of any six consecutive terms; also represent the scale of relation by m, n, r. Then we have

$$\left.\begin{aligned} ma + nb + rc &= d \\ mb + nc + rd &= e \\ mc + nd + re &= f \end{aligned}\right\} \quad \ldots\ldots \quad (T).$$

These three equations will determine the values of m, n, and r.

2d. To find the sum of the series, represent the terms by A, B, C, etc., and put

$$S = A + B + C + D + E + F + \ldots\ldots \quad (1).$$

Then because the series is of the third order we shall have

$$\left.\begin{aligned} D &= mAx^3 + nBx^2 + rCx \\ E &= mBx^3 + nCx^2 + rDx \\ F &= mCx^3 + nDx^2 + rEx \\ \text{etc.,} &\quad \text{etc.,} \quad \text{etc.,} \quad \text{etc.} \end{aligned}\right\} \quad \ldots \quad (2).$$

By addition, observing the value of S in (1), we have

$$S - A - B - C = mx^3S + nx^2(S - A) + rx(S - A - B);$$

whence,

$$S = \frac{A + B + C - (A + B)rx - Anx^2}{1 - rx - nx^2 - mx^3} \quad . \quad (V).$$

In like manner formulas may be obtained for the summation of recurring series of higher orders.

391. To apply these formulas in the summation of any given series, we must first determine the scale of relation by (P) or (T), and then we may obtain the sum of the series from (Q) or (V).

If the order of the series is not known, we should first determine the values of m and n by formula (P), and ascertain by trial whether the scale of relation thus found will apply to the given series. If it will not apply, we may determine the values of m, n, and r from formula (T), and ascertain by trial whether the series can be developed by the new scale thus obtained. If this also fail, we must establish other formulas corresponding to still higher degrees, and continue the trials.

If, however, we resort in the first place to a formula corresponding to an order higher than that of the given series, then one or more of the quantities, m, n, r, etc., will prove to be zero, and the remaining numbers may be taken as the scale of relation, without further trial.

1. Find the sum of the infinite series, $1 + 4x + 10x^2 + 22x^3 + 46x^4 + \ldots.$

To determine the scale of relation, we have

$$a = 1, \ b = 4, \ c = 10, \ d = 22.$$

These values substituted in formula (P), give

$$m + 4n = 10,$$
$$4m + 10n = 22,$$

from which we readily obtain

$$m = -2, \ n = 3.$$

These numbers form the true scale of relation; for we perceive that any coefficient after the second in the given series, is equal to three times the first preceding coefficient, minus twice the second preceding coefficient.

To find the sum of the series, we have

$$A = 1, \quad B = 4x.$$

Whence by formula (Q),

$$S = \frac{1 + 4x - 3x}{1 - 3x + 2x^2} = \frac{1 + x}{1 - 3x + 2x^2}, \ \textit{Ans.}$$

We have thus obtained the sum of the series in the form of an

algebraic fraction. Conversely, the given series may be developed from this fraction, either by division, or by the method of indeterminate coefficients. Indeed, it will be found that the sum of every recurring series is an irreducible fraction, from which the series may be supposed to originate. The fraction from which any particular series is supposed to arise, is called the *generating fraction* for that series; it is the same as the sum of the series.

EXAMPLES FOR PRACTICE.

1. Find the sum of $1 + 3x + 4x^2 + 7x^3 + 11x^4 + \ldots.$

Ans. $\dfrac{1+2x}{1-x-x^2}.$

2. Find the sum of $1 + 6x + 12x^2 + 48x^3 + 120x^4 + \ldots.$

Ans. $\dfrac{1+5x}{1-x-6x^2}.$

3. Find the sum of $1 + 2x - 5x^2 + 26x^3 - 119x^4 + \ldots.$

Ans. $\dfrac{1+6x}{1+4x-3x^2}.$

4. Find the sum of $1+4x+3x^2-2x^3+4x^4+17x^5+3x^6+\ldots.$

Ans. $\dfrac{1+3x+x^2}{1-x+2x^2-3x^3}.$

5. Find the sum of $1 + 3x + 5x^2 + 7x^3 + 9x^4 + \ldots.$

Ans. $\dfrac{1+x}{(1-x)^2}.$

6. Find the sum of $1 + x + 5x^2 + 13x^3 + 41x^4 + 121x^5 + \ldots.$

Ans. $\dfrac{1-x}{1-2x-3x^2}.$

7. Find the sum of $1 + 4x + 6x^2 + 11x^3 + 28x^4 + 63x^5 + \ldots.$

Ans. $\dfrac{(1+x)^2-2x^3}{(1-x)^2-3x^3}.$

8. Find the sum of $\dfrac{x}{2} + x^3 + \dfrac{7x^5}{2} + 10x^7 + \dfrac{61x^9}{2} + 91x^{11} + \ldots.$

Ans. $\dfrac{x}{2-4x^2-6x^4}.$

DIFFERENTIAL METHOD.

392. The ***Differential Method*** is the process of finding any term of a regular series, or the sum of any number of terms, by means of the *successive differences* of the terms.

393. To find any term of a series by the differential method.

If we subtract each term of a series from the next succeeding term, we shall obtain a new series called the *first order of differences.* If we subtract each term of this new series from the succeeding term, we shall obtain a series called the *second order of differences;* and so on.

Let $a, b, c, d, e, \ldots.$ represent a regular series, the successive terms being formed according to any fixed law. We will write the given terms in a vertical column, and proceed by actual subtraction to form the several orders of differences, placing each order in a separate column, and each difference at the right of the subtrahend. The result is as follows:

Series.	1st order of differences.	2d order of differences.	3d order of differences.	4th order of differences.
a				
b	$b-a$			
c	$c-b$	$c-2b+a$		
d	$d-c$	$d-2c+b$	$d-3c+3b-a$	
e	$e-d$	$e-2d+c$	$e-3d+3c-b$	$e-4d+6c-4b+a$

The quantity which stands first in any column, though a polynomial, is called the *first term* of the order of differences which designates the column.

Let d_1, d_2, d_3, d_4, etc., represent the *first terms* of the first, second, third, fourth, etc., orders of differences. Then we shall have

$$d_1 = b - a,$$
$$d_2 = c - 2b + a,$$
$$d_3 = d - 3c + 3b - a,$$
$$d_4 = e - 4d + 6c - 4b + a, \text{ etc.}$$

By transposition, we may obtain, after making the necessary substitutions,

$$
\begin{aligned}
a &= a,\\
b &= a + d_1,\\
c &= a + 2d_1 + d_2,\\
d &= a + 3d_1 + 3d_2 + d_3,\\
e &= a + 4d_1 + 6d_2 + 4d_3 + d_4, \text{ etc.}
\end{aligned}
$$

These equations express the values of a, b, c, d, e, etc., in terms of a, d_1, d_2, d_3, d_4, etc. The coefficients in the second members are formed according to the *binomial formula;* and we observe that the coefficients of the second power of a binomial are found in the third equation, the coefficients of the third power in the fourth equation, and so on.

Hence, if we let T_{n+1} denote the $(n+1)th$ term of the given series,

$$a, b, c, d, e, \ldots .$$

then we shall have

$$T_{n+1} = a + nd_1 + \frac{n(n-1)}{2} d_2 + \frac{n(n-1)(n-2)}{2 \cdot 3} d_3 + \ldots . \quad (m).$$

And by substituting $n-1$ for n, we shall obtain a formula for the n^{th} term of the given series; thus,

$$T_n = a + (n-1)d_1 + \frac{(n-1)(n-2)}{2} d_2 + \frac{(n-1)(n-2)(n-3)}{2 \cdot 3} d_3 \ldots . \quad (A).$$

394. To find the sum of any number of terms of a series, by the differential method.

Represent the given series by

$$a, b, c, d, e, \ldots . \quad (1).$$

And denote the sum of n terms by S. We are to find the values of S in functions of

$$a, d_1, d_2, d_3, \text{ etc.}$$

Let us assume the auxiliary series,

$$0,\ a,\ a+b,\ a+b+c,\ a+b+c+d, \ldots . \quad (2).$$

It is obvious that the $(n+1)th$ term of this series is the same as the sum of n terms of the given series (1), and may be placed equal to S. Now let

$$d'_1,\ d'_2,\ d'_3,\ d'_4, \text{ etc.,}$$

represent the first terms of the successive orders of differences in the auxiliary series (2). Then by formula (m), we have

$$S = 0 + nd'_1 + \frac{n(n-1)}{2} d'_2 + \frac{n(n-1)(n-2)}{2 \cdot 3} d'_3 + \dots \quad (n).$$

But if we proceed to form the first, second, third, etc., orders of differences for the auxiliary series (2), we shall have

$$d'_1 = a,$$
$$d'_2 = b - a = d_1,$$
$$d'_3 = c - 2b + a = d_2, \text{ etc.}$$

Hence, by substitution in equation (n), we have

$$S = na + \frac{n(n-1)}{2} d_1 + \frac{n(n-1)(n-2)}{2 \cdot 3} d_2 + \dots \quad (B).$$

395. The use of formulas (A) and (B) may be illustrated by the following examples:

1. Find the 12th term of the series, 1, 5, 15, 35, 70, 126, etc.

We first form the successive orders of differences, as follows:

1,					
5,	4,				
15,	10,	6,			
35,	20,	10,	4,		
70,	35,	15,	5,	1,	
126,	56,	21,	6,	1,	0.

Thus we have $n = 12$, and

$$a = 1, \quad d_1 = 4, \quad d_2 = 6, \quad d_3 = 4, \quad d_4 = 1, \quad d_5 = 0.$$

Substituting these values in (A), and reducing the terms, we obtain

$$T_{12} = 1 + 44 + 330 + 660 + 330 = 1365, \textit{ Ans.}$$

The series is broken off at the fifth term, because the subsequent differences are all zero.

2. Sum the series 1, 3, 6, 10, 15, 21, etc., to n terms.

By forming the successive orders of differences, as in the last example, we shall obtain

$$a = 1, \ d_1 = 2, \ d_2 = 1, \ d_3 = 0.$$

Whence, by formula (B),

$$S = n + \frac{n(n-1)}{2} \cdot 2 + \frac{n(n-1)(n-2)}{2 \cdot 3},$$

all the terms after the third becoming zero. By performing the indicated operations, adding the results, and then factoring, we have

$$S = \frac{n(n+1)(n+2)}{6}, \textit{ Ans.}$$

EXAMPLES FOR PRACTICE.

1. Find the 9th term of the series 1, 4, 8, 13, 19, etc.
Ans. 53.

2. Find the 15th term of the series 1, 4, 10, 20, 35, etc.
Ans. 680.

3. Find the 8th and 9th terms of the series 1, 6, 21, 56, 126, 251, 456, etc. *Ans.* 771 and 1231.

4. Find the 20th term of the series 1, 8, 27, 64, 125, etc.
Ans. 8000.

5. Find the n^{th} term of the series 1, 3, 6, 10, 15, 21, etc.
Ans. $\frac{n(n+1)}{2}$.

6. Find the n^{th} term of the series 1, 4, 10, 20, 35, etc.
Ans. $\frac{n(n+1)(n+2)}{6}$.

7. Find the n^{th} term of the series 1, 5, 15, 35, 70, 126, etc.
Ans. $\frac{n(n+1)(n+2)(n+3)}{24}$.

8. Sum the series 1, 3, 6, 10, 15, 21, etc., to 20 terms.
Ans. 1540.

9. Sum the series 1, 5, 14, 30, 55, 91, etc., to 12 terms.
Ans. 2366.

10. Sum the series 1, 4, 13, 37, 85, 166, etc., to 10 terms.
Ans. 2755.

11. Sum the series $1\cdot2$, $2\cdot3$, $3\cdot4$, $4\cdot5$, $5\cdot6$, etc., to n terms.
Ans. $\frac{n(n+1)(n+2)}{3}$.

12. Sum the series 1·2·3, 2·3·4, 3·4·5, 4·5·6, etc., to n terms. *Ans.* $\frac{n(n+1)(n+2)(n+3)}{4}$.

13. Sum the series 1^2, 2^2, 3^2, 4^2, 5^2, etc., to n terms.

Ans. $\frac{n(n+1)(2n+1)}{6}$.

14. Sum the series 1^3, 2^3, 3^3, 4^3, 5^3, etc., to n terms.

Ans. $\frac{(n^2+n)^2}{4}$.

15. Sum the series 1^4, 2^4, 3^4, 4^4, 5^4, etc., to n terms.

Ans. $\frac{n^5}{5}+\frac{n^4}{2}+\frac{n^3}{3}-\frac{n}{30}$.

16. Sum the series $(m+1)$, $2(m+2)$, $3(m+3)$, $4(m+4)$, etc., to n terms. *Ans.* $\frac{n(n+1)(1+2n+3m)}{6}$.

INTERPOLATION.

396. ***Interpolation*** is the process of introducing between two consecutive terms of a series, a term or terms which shall conform to the law of the series. It is of great use in the construction of mathematical tables, and in the calculations of Astronomy.

397. The interpolation of terms in a series is effected by the differential method. In any series, the value of a term which has n terms before it, is expressed by formula (m) (**393**), which is

$$T_{1+n} = a + nd_1 + \frac{n(n-1)}{2}d_2 + \frac{n(n-1)(n-2)}{2\cdot 3}d_3 + \ldots$$

If in this formula we make n a *fraction*, then the resulting equation will give the value of a term intermediate between two of the given terms, and related to the others by the law of the series.

If n is less than unity, the intermediate term will lie between the first and second of the given terms; if n is greater than 1 and less than 2, the intermediate term will lie between the second and third of the given terms; and so on.

Given $\left\{\begin{array}{l} \sqrt[3]{21} = 2.758924 \\ \sqrt[3]{22} = 2.802039 \\ \sqrt[3]{23} = 2.843867 \\ \sqrt[3]{24} = 2.884499 \\ \sqrt[3]{25} = 2.924018 \end{array}\right\}$ to find the cube roots of intermediate numbers, by interpolation.

1. Required the cube root of 21.75.

We have

No.	Cube Roots.	d_1	d_2	d_3	d_4
21	2.758924				
22	2.802039	+.043115			
23	2.843867	+.041828	−.001287		
24	2.884499	+.040632	−.001196	+.000091	
25	2.924018	+.039519	−.001113	+.000083	−.000008

Hence, to find the cube root of 21.75 by the formula, we have $a = 2.758924$, $n = .75$,

$$d_1 = +.043115, \quad d_2 = -.001287, \quad d_3 = +.000091, \text{ etc.}$$

These values substituted in the formula, give

1st	term,	+ 2.758924
2d	"	+ .032336
3d	"	+ .000121
4th	"	+ .000004
Whence		2.791385, *Ans.*

If it were required to find the cube root of any number between 22 and 23, we might put n equal to the excess of the number above 21, and employ the same values for d_1, d_2, d_3, etc., as before. But greater accuracy will be attained by making 22 the first term of the series, and employing the corresponding differences; in which case n will be a proper fraction.

EXAMPLES FOR PRACTICE.

Find by interpolation,

1. The cube root of 21.325. *Ans.* 2.773083.

2. The cube root of 21.875. *Ans.* 2.796722.

3. The cube root of 21.4568. *Ans.* 2.778785.

4. The cube root of 22.25. *Ans.* 2.812613.

5. The cube root of 22.684. *Ans.* 2.830783.

6. The cube root of 22.75. *Ans.* 2.833525.

398. On three successive days, the angular distances of the sun from the moon, as seen from the earth, were as follows:

1st	day,	noon,	66° 6′ 38″.
"	"	midnight,	72° 24′ 5″.
2d	"	noon,	78° 34′ 48″.
"	"	midnight,	84° 39′ 4″.
3d	"	noon,	90° 37′ 18″.
"	"	midnight,	96° 29′ 57″.

In the data here given, the interval of time is 12 hours. Hence, to find the distance of the sun from the moon at intermediate times, n must always be some fractional part of 12. Thus, for the distance at 3 o'clock P. M. of the first day we have $n=\frac{3}{12}=\frac{1}{4}$, and $a = 66° 6' 38''$; for the distance at 6 o'clock A. M. of the second day, $n=\frac{6}{12}=\frac{1}{2}$, and $a = 72° 24' 5''$. For the distance at 3 o'clock P. M. of the second day, $n=\frac{3}{12}=\frac{1}{4}$, and $a = 78° 34' 48''$.

EXAMPLES FOR PRACTICE.

Find by interpolation the distance of the sun from the moon,

1. At 3 o'clock P. M. of the first day. *Ans.* 67° 41′ 39″.
2. At 6 o'clock P. M. of the first day. *Ans.* 69° 16′ 13″.
3. At 9 o'clock P. M. of the first day. *Ans.* 70° 50′ 22″.
4. At 3 o'clock A. M. of the second day. *Ans.* 73° 57′ 23″.
5. At 6 o'clock A. M. of the second day. *Ans.* 75° 30′ 16″.
6. At 9 o'clock A. M. of the second day. *Ans.* 77° 2′ 44″.
7. At 3 o'clock P. M. of the second day. *Ans.* 80° 6′ 27″.
8. At 6 o'clock P. M. of the second day. *Ans.* 81° 37′ 43″.
9. At 9 o'clock P. M. of the second day. *Ans.* 83° 8′ 35″.

LOGARITHMS.

399. The ***Logarithm*** of a number is the exponent of the power to which a certain other number, called the *base*, must be raised, in order to produce the given number. Thus, in the expression,

$$a^x = b,$$

the exponent, x, is the logarithm of b to the base a.

An equation in this form is called an *exponential equation*.

If in this equation we suppose a to be constant, while b is made equal to every possible positive number in succession, the corresponding values of x will constitute a *system* of logarithms; hence,

400. A ***System of Logarithms*** consists of the logarithms of all possible positive numbers, according to a given base.

Any positive number except unity may be made the base of a system of logarithms. For, by giving to x suitable values, the equation

$$a^x = b$$

will be true for all possible positive values of b, provided a is any positive number except 1. Hence,

There may be an indefinite number of systems of logarithms.

401. If in the equation $a^x = b$, we suppose b to represent a perfect power of a, then x will be some *integer;* but if b is not a perfect power of a, then x will be some *fraction*. Hence,

A logarithm may consist of an integral and a fractional part.

402. The ***Index*** or ***Characteristic*** of a logarithm is the integral part; and

403. The ***Mantissa*** is the fractional part of a logarithm.

For illustration, let 5 be the base of a system; then we have

$$5^{2.25} = 5^{\frac{9}{4}} = \sqrt[4]{5^9} = 37.384.$$

Thus, the logarithm of 37.384 to the base 5, is 2.25; the *index* of this logarithm is 2, and the *mantissa* .25.

PROPERTIES OF LOGARITHMS.

404. There are certain properties of logarithms, which are common to all systems. To investigate these general properties,

let a denote the base of the system; also, designate the logarithm of a quantity by log., written before the quantity.

1. *In any system, the logarithm of unity is* 0.

For, let $a^x = 1$; then $x = \log. 1$.

But by **88**, if $a^x = 1$, then $x = 0$, or $\log. 1 = 0$.

2. *In any system, the logarithm of the base is unity.*

For, let $a^x = a$; then $x = \log. a$.

But by **88**, if $a^x = a$, then $x = 1$, or $\log. a = 1$.

3. *The logarithm of the product of two numbers is equal to the sum of the logarithms of the two numbers.*

For, let $m = a^x$, $n = a^z$;

then $x = \log. m$, $z = \log. n$.

But by multiplication, $mn = a^{x+z}$;

therefore, $\log. mn = x + z = \log. m + \log. n$.

4. *The logarithm of a quotient is equal to the logarithm of the dividend diminished by the logarithm of the divisor.*

For, let $m = a^x$, $n = a^z$;

then $x = \log. m$, $z = \log. n$.

By division, $\frac{m}{n} = a^{x-z}$;

therefore, $\log. \left(\frac{m}{n}\right) = x - z = \log. m - \log. n$.

5. *The logarithm of any power of a number is equal to the logarithm of the number multiplied by the exponent of the power.*

For, let $m = a^x$; then $x = \log. m$.

By involution, $m^r = a^{rx}$;

therefore, $\log. (m^r) = rx = r \log. m$.

6. *The logarithm of any root of a number is equal to the logarithm of the number divided by the index of the root.*

For, let $m = a^x$; then $x = \log. m$.

By evolution, $\sqrt[r]{m} = a^{\frac{x}{r}}$;

therefore, $\log. \sqrt[r]{m} = \frac{x}{r} = \frac{\log. m}{r}$.

405. The principal use of logarithms is to facilitate arithmetical computations. By means of the last four properties, we may avoid the ordinary labor of multiplication, division, involution, and evolution,—these operations being practically performed by *addition* and *subtraction.*

For this purpose, it is necessary to have a *Table of Logarithms,* so constructed that we may readily obtain the logarithm of any number within a certain limit, or the number corresponding to any logarithm, to a certain degree of approximation. The common tables give the logarithms of numbers from 1 to 10,000, correct to 6 decimal places.

With a table of this kind, we have the following obvious

RULES FOR COMPUTATION.

I. To multiply one number by another:—*Find the logarithms of the given numbers; add these logarithms, and find the number corresponding to the sum; this number will be the required product* (**404**, 3).

II. To divide one number by another:—*Find the logarithms of the given numbers; subtract the logarithm of the divisor from that of the dividend, and find the number corresponding to the difference; this number will be the required quotient* (**404**, 4).

III. To raise a number to any power:—*Find the logarithm of the given number, and multiply it by the exponent of the required power; then find the number corresponding to this product, and it will be the required power* (**404**, 5).

IV. To extract any root of a number:—*Find the logarithm of the given number, and divide it by the index of the root; then find the number corresponding to the quotient, and it will be the required root* (**404**, 6).

NOTE.—From **400**, we infer that *negative numbers, as such, have no logarithms.* But we may always employ logarithms in calculations where negative factors are involved, by disregarding signs until the absolute value of the product or quotient is obtained.

THE COMMON SYSTEM.

406. Any positive number greater than unity may be made the base of a system of logarithms. But the only base used in practical calculations, is 10. The logarithms of numbers according to this base, form what is called the *Common System* of logarithms.

NOTE.—Besides the common system, there is another, called the *Naperian System*, from Baron Napier, the inventor of logarithms. This system is of great theoretical importance, and its relation to other systems will be shown in a subsequent article.

407. The peculiarities which constitute the advantage of the common system, may be shown as follows:

Since 10 is the base of the system,

$$\begin{aligned} \log.\ 1 &= \log.\ 10^0 = 0, \\ \log.\ 10 &= \log.\ 10^1 = 1, \\ \log.\ 100 &= \log.\ 10^2 = 2, \\ \log.\ 1000 &= \log.\ 10^3 = 3, \\ \log.\ 10000 &= \log.\ 10^4 = 4. \end{aligned}$$

Now it is obvious that if any number, integral or mixed, be greater than 1 and less than 10, its logarithm will be entirely decimal; if the number be greater than 10 and less than 100, its logarithm will be 1 plus a decimal; if greater than 100 and less than 1000, its logarithm will be 2 plus a decimal; and so on. Hence,

1. *The common logarithm of an integer or a mixed number will have a positive index, equal to the number of integral places minus* 1.

Again, since the logarithm of 10 is 1, it follows that if a number be divided by 10 continually, the logarithm will be diminished by 1 continually, the decimal part remaining unchanged.

Let us take any number, as 5468, and denote the mantissa, or the decimal part of its logarithm, by m. Then we have

(1.)	(2).
$\log.\ 5468 = 3+m,$	$\log.\ .5468 = -1+m,$
$\log.\ 546.8 = 2+m,$	$\log.\ .05468 = -2+m,$
$\log.\ 54.68 = 1+m,$	$\log.\ .005468 = -3+m,$
$\log.\ 5.468 = 0+m;$	$\log.\ .0005468 = -4+m;$

in which 3, 2, 1, 0, are the indices of the logarithms in column (1); and -1, -2, -3, -4, are the indices of the logarithms in column (2); and m, the decimal part in all. Hence,

2. *If two numbers consist of the same figures, and differ only in the position of the decimal point, their logarithms, in the common system, will have the same decimal part, and will differ only in the values of the index.*

3. *The common logarithm of a decimal fraction will have a negative index; if the significant part of the decimal commence at the tenths' place, the index of the logarithm will be* -1; *but if ciphers occur between the decimal point and the first significant figure, the index of the logarithm will be numerically equal to the number of intervening ciphers, plus* 1.

408. In writing the logarithm of a decimal fraction, the minus sign is placed before the index, and the decimal or positive part annexed without any intervening sign. Thus, from a table of logarithms, we have

$$\log.\ .0546 = -2.737193,$$

in which the minus sign must be understood as affecting only the index 2. This logarithm is therefore equivalent to

$$-2 + .737193.$$

COMPUTATION OF LOGARITHMS.

409. Since the rules for computing by logarithms require a logarithmic table, it becomes necessary to calculate the logarithms of an extended series of numbers. The only practical method of doing this, is by means of a converging series, expressing the value of any logarithm in known terms.

Let us resume the fundamental equation,

$$a^x = b \quad . \quad . \quad . \quad . \quad . \quad (1),$$

in which x is the logarithm of b, to the base a.

Assume $\qquad a = 1 + c, \quad b = 1 + p;$

then $\qquad (1 + c)^x = 1 + p \quad . \quad . \quad . \quad (2),$

where x is the logarithm of $1 + p$, to the base a.

Raise both members of equation (2) to the n^{th} power; then

$$(1+c)^{nx} = (1+p)^n.$$

Expanding both members by the Binomial Theorem, we have

$$1 + nxc + \frac{nx(nx-1)}{2}c^2 + \frac{nx(nx-1)(nx-2)}{2\cdot 3}c^3 + \frac{nx(nx-1)(nx-2)(nx-3)}{2\cdot 3\cdot 4}c^4 + \ldots = 1 + np + \frac{n(n-1)}{2}p^2 + \frac{n(n-1)(n-2)}{2\cdot 3}p^3 + \frac{n(n-1)(n-2)(n-3)}{2\cdot 3\cdot 4}p^4 + \ldots$$

Dropping unity from both members, and dividing by n, we obtain

$$x\left(c + \frac{(nx-1)}{2}c^2 + \frac{(nx-1)(nx-2)}{2\cdot 3}c^3 + \frac{(nx-1)(nx-2)(nx-3)}{2\cdot 3\cdot 4}c^4 + \ldots\right)$$

$$= p + \frac{(n-1)}{2}p^2 + \frac{(n-1)(n-2)}{2\cdot 3}p^3 + \frac{(n-1)(n-2)(n-3)}{2\cdot 3\cdot 4}p^4 + \ldots$$

This equation is true for all values of n; it will be true, therefore, when $n=0$. Making this supposition, the equation reduces to

$$x\left(c - \frac{c^2}{2} + \frac{c^3}{3} - \frac{c^4}{4} + \frac{c^5}{5} - \ldots\right) = p - \frac{p^2}{2} + \frac{p^3}{3} - \frac{p^4}{4} + \frac{p^5}{5} - \ldots \quad . \ . \ . \ (3).$$

From (2) we perceive that

$$x = \log.\ (1+p).$$

Hence, if we place

$$M = \frac{1}{c - \frac{1}{2}c^2 + \frac{1}{3}c^3 - \frac{1}{4}c^4 + \frac{1}{5}c^5 - \ldots} \quad . \ . \ . \ (4),$$

(3) will become

$$\log.\ (1+p) = M\left(p - \frac{p^2}{2} + \frac{p^3}{3} - \frac{p^4}{4} + \frac{p^5}{5} - \ldots\right) \quad . \ . \ . \ (A).$$

Thus, we have obtained an expression for the logarithm of the number $1+p$, or b. This expression consists of two factors; namely, the quantity in the parenthesis, which depends upon the number, and the quantity M, which depends upon the base of the system.

410. It is obvious that if a definite value be given to M, the base of the system will be fixed and determinate. Baron Napier arbitrarily assumed $\qquad M = 1.$

To determine the base of the system, according to this assumption, substitute 1 for M in equation (4) (**409**). We shall have, after reducing,

$$1 = c - \frac{c^2}{2} + \frac{c^3}{3} - \frac{c^4}{4} + \frac{c^5}{5} - \ldots.$$

Putting $s = 1$, we shall have

$$s = c - \frac{c^2}{2} + \frac{c^3}{3} - \frac{c^4}{4} + \frac{c^5}{5} - \ldots.$$

Reverting the series, we obtain

$$c = s + \frac{s^2}{1\cdot 2} + \frac{s^3}{1\cdot 2\cdot 3} + \frac{s^4}{1\cdot 2\cdot 3\cdot 4} + \frac{s^5}{1\cdot 2\cdot 3\cdot 4\cdot 5} + \ldots.$$

Restoring the value of s,

$$c = 1 + \frac{1}{1\cdot 2} + \frac{1}{1\cdot 2\cdot 3} + \frac{1}{1\cdot 2\cdot 3\cdot 4} + \frac{1}{1\cdot 2\cdot 3\cdot 4\cdot 5} + \ldots.$$

By taking 12 terms of this series, we find the approximate value of c to be 1.7182818. But the base is $1 + c$; hence, adding 1 to the result, and representing the sum by e, the usual symbol for the Naperian base, we have

$$e = 2.7182818,$$

which is the base of the Naperian system.

411. In the general formula, (A), the quantity M, which depends upon the base, is called the *modulus* of the system. Thus, the modulus of the Naperian system is unity.

Let us here designate Naperian logarithms by nap. log., and logarithms in any other system by log., simply. Then,

$$\text{log.}\,(1+p) = M\left(p - \frac{p^2}{2} + \frac{p^3}{3} - \frac{p^4}{4} + \frac{p^5}{5} - \ldots\right) \;.\;.\;. \quad (1),$$

$$\text{nap. log.}\,(1+p) = \left(p - \frac{p^2}{2} + \frac{p^3}{3} - \frac{p^4}{4} + \frac{p^5}{5} - \ldots\right) \;.\;.\;. \quad (2).$$

Dividing (1) by (2), we obtain

$$M = \frac{\text{log.}\,(1+p)}{\text{nap. log.}\,(1+p)} \;.\;.\;.\;.\;.\;.\;.\;. \quad (3);$$

or, $$\{\text{nap. log.}\,(1+p)\} \times M = \text{log.}\,(1+p) \;.\;.\;. \quad (4),$$

where M is the modulus of the system in which the logarithm of the second member is taken. Hence,

The modulus of any particular system is the constant multiplier which will convert Naperian logarithms into the logarithms of that system.

412. Formula (A) can be employed for the computation of logarithms, only when p is less than unity; for if p be greater than unity, the series will not converge. The series, however, may be transformed into another which will always *converge*.

Let us resume the logarithmic series,

$$\log.(1+p) = M\left(p - \frac{p^2}{2} + \frac{p^3}{3} - \frac{p^4}{4} + \frac{p^5}{5} - \dots\right) \quad . \quad . \quad . \quad (1).$$

If in this equation we substitute $-p$ for p, we shall have

$$\log.(1-p) = M\left(-p - \frac{p^2}{2} - \frac{p^3}{3} - \frac{p^4}{4} - \frac{p^5}{5} - \dots\right) \quad . \quad . \quad . \quad (2).$$

If we subtract (2) from (1), observing that

$$\log.(1+p) - \log.(1-p) = \log.\left(\frac{1+p}{1-p}\right),$$

we shall have

$$\log.\left(\frac{1+p}{1-p}\right) = 2M\left(p + \frac{p^3}{3} + \frac{p^5}{5} + \frac{p^7}{7} + \dots\right) \quad . \quad . \quad . \quad (3).$$

Assume $\quad p = \dfrac{1}{2z+1}$; whence, $\dfrac{1+p}{1-p} = \dfrac{z+1}{z}$.

These values substituted in (3), give

$$\log.\left(\frac{z+1}{z}\right) = 2M\left(\frac{1}{2z+1} + \frac{1}{3(2z+1)^3} + \frac{1}{5(2z+1)^5} + \frac{1}{7(2z+1)^7} + \dots\right) \quad . \quad . \quad . \quad (4).$$

The first member of this equation is equivalent to $\log.(z+1) - \log.z$. Hence, finally, we have

$$\log.(z+1) - \log.z =$$
$$2M\left(\frac{1}{2z+1} + \frac{1}{3(2z+1)^3} + \frac{1}{5(2z+1)^5} + \frac{1}{7(2z+1)^7} + \dots\right) \quad . \quad . \quad . \quad (B).$$

This series is rapidly converging, and may be employed with facility for the computation of logarithms, in the Naperian, or in the common system.

To commence the construction of a table, first make $z = 1$; then $\log. z = 0$, and the formula will give the value of $\log.(z+1)$, or log. 2. Next make $z = 2$; then the formula will give the value of $\log.(z+1)$, or log. 3; and so on.

It is necessary to compute directly the logarithms of *prime* numbers only, in any system; for, according to **404**, 3, the logarithm of any composite number may be obtained, by adding the logarithms of its factors.

413. We will now illustrate the use of formula (B), by computing the *Naperian* logarithms of 2, 4, 5, and 10.

Make $z = 1$; then nap. log. $z = 0$, and nap. log. $(z + 1) =$ nap. log. 2; and since $M = 1$, we have

$$\text{nap. log. } 2 = 2\left(\frac{1}{1 \cdot 3} + \frac{1}{3 \cdot 3^3} + \frac{1}{5 \cdot 3^5} + \frac{1}{7 \cdot 3^7} + \dots\right).$$

Form a column of numbers consisting of $\frac{2}{3}$ and the quotients obtained by dividing $\frac{2}{3}$ by 3^2, or 9, continually; then dividing the first of these numbers by 1, the second by 3, the third by 5, and so on, we obtain the several terms of the series.

$$\begin{array}{r|rcrcr}
3 & 2 & & & & \\
9 & 0.66666666 & \div & 1 & = & .66666666 \\
9 & 7407407 & \div & 3 & = & 2469136 \\
9 & 823045 & \div & 5 & = & 164609 \\
9 & 91449 & \div & 7 & = & 13064 \\
9 & 10161 & \div & 9 & = & 1129 \\
9 & 1129 & \div & 11 & = & 103 \\
9 & 125 & \div & 13 & = & 10 \\
 & 14 & \div & 15 & = & 1 \\
\hline
 & & & & & .69314718 = \text{nap. log. } 2. \\
 & & & & & 2 \\
\hline
\end{array}$$

Whence, by **404**, 5, $\quad 1.38629436 = \text{nap. log. } 4.$

Next make $z = 4$; then $z + 1 = 5$; and $2z + 1 = 9$; and we have

$$\text{nap. log. } 5 = 2\left(\frac{1}{1 \cdot 9} + \frac{1}{3 \cdot 9^3} + \frac{1}{5 \cdot 9^5} + \frac{1}{7 \cdot 9^7} + \dots\right) + \text{nap. log. } 4.$$

$$\begin{array}{r|rcrcr}
9 & 2 & & & & \\
9^2 = 81 & 0.22222222 & \div & 1 & = & .22222222 \\
81 & 274348 & \div & 3 & = & 91449 \\
81 & 3387 & \div & 5 & = & 677 \\
 & 42 & \div & 7 & = & 6 \\
\hline
 & & & & & .22314354, \text{ sum of series.}
\end{array}$$

$$\begin{array}{llr} \text{To} & & .22314354 \\ \text{Add} & \text{nap. log. } 4 = & \underline{1.38629436} \\ & & 1.60943790 = \text{nap. log. } 5. \\ \text{Add} & \text{nap. log. } 2 = & \underline{.69314718} \\ \text{Whence, by } \mathbf{404}, 3, & & 2.30258508 = \text{nap. log. } 10. \end{array}$$

414. In order to compute *common* logarithms, we must first determine the modulus of the common system. From **411**, equation (3), we have

$$M = \frac{\log. (1 + p)}{\text{nap. log. } (1 + p)}.$$

In this equation, make $1 + p = 10$, the base of the common system. Then we have

$$M = \frac{1}{2.30258508} = .43429448 \quad . \quad . \quad . \quad (1),$$

the value of the modulus sought. Substituting this value in formula (B), we obtain the formula for common logarithms, as follows:

$$\log. (z + 1) - \log. z =$$

$$.86858896\left(\frac{1}{2z+1} + \frac{1}{3(2z+1)^3} + \frac{1}{5(2z+1)^5} + \frac{1}{7(2z+1)^7} + \ldots.\right)\ldots(C).$$

To apply this formula, assume $z = 10$; then

$$\log. z = 1, \quad \text{and} \quad 2z + 1 = 21.$$

$$\begin{array}{r|lr} 21 & .86858896 & \\ 21^2 = 441 & .04136138 \div 1 = & .04136138 \\ 441 & \quad\quad 9379 \div 3 = & 3126 \\ & \quad\quad\quad 21 \div 5 = & \underline{4} \\ & & .04139268, \text{ sum of series.} \end{array}$$

$$\begin{array}{lr} \text{Add} \quad \log. z = & \underline{1.0} \\ \log. (z + 1) = & 1.04139268 = \log. 11. \end{array}$$

If we make $z = 99$, then $z + 1 = 100$, and $2z + 1 = 199$.
In this case, the formula will give the logarithm of 99; for,

$$\log. (z + 1) - \log. z = \log. 100 - \log. 99 = 2 - \log. 99.$$

$$\begin{array}{r|lr} 199 & .86858896 & \\ 199^2 = 39601 & \quad 436477 \div 1 = & .00436477 \\ & \quad\quad\quad 11 \div 3 = & \underline{4} \\ & & .00436481, \text{ sum of series.} \end{array}$$

Therefore, we have

$$2 - \log.\ 99 = .00436481$$

whence $\qquad 1.99563519 = \log.\ 99.$

Subtract $\qquad \log.\ 11 = \underline{1.04139268}$

$$.95424251 = \log.\ 9.$$

And by **404**, 6, $\qquad \tfrac{1}{2} \log.\ 9 = .47712126 = \log.\ 3.$

Thus we may compute logarithms with great facility, using the formula for prime numbers only.

USE OF TABLES.

415. The following contracted tables will illustrate the principles of logarithms, and the methods of using the larger tables. The logarithms are taken in the common system.

TABLE I.—LOGARITHMS FROM 1 TO 100.

N.	LOG.	N.	LOG.	N.	LOG.	N.	LOG.
1	0 000000	26	1 414973	51	1 707570	76	1 880814
2	0 301030	27	1 431364	52	1 716003	77	1 886491
3	0 477121	28	1 447158	53	1 724276	78	1 892095
4	0 602060	29	1 462398	54	1 732394	79	1 897627
5	0 698970	30	1 477121	55	1 740363	80	1 903090
6	0 778151	31	1 491362	56	1 748188	81	1 908485
7	0 845098	32	1 505150	57	1 755875	82	1 913814
8	0 903090	33	1 518514	58	1 763428	83	1 919078
9	0 954243	34	1 531479	59	1 770852	84	1 924279
10	1 000000	35	1 544068	60	1 778151	85	1 929419
11	1 041393	36	1 556303	61	1 785330	86	1 934498
12	1 079181	37	1 568202	62	1 792392	87	1 939519
13	1 113943	38	1 579784	63	1 799341	88	1 944483
14	1 146128	39	1 591065	64	1 806180	89	1 949390
15	1 176091	40	1 602060	65	1 812913	90	1 954243
16	1 204120	41	1 612784	66	1 819544	91	1 959041
17	1 230449	42	1 623249	67	1 826075	92	1 963788
18	1 255273	43	1 633468	68	1 832509	93	1 968483
19	1 278754	44	1 643453	69	1 838849	94	1 973128
20	1 301030	45	1 653213	70	1 845098	95	1 977724
21	1 322219	46	1 662758	71	1 851258	96	1 982271
22	1 342423	47	1 672098	72	1 857333	97	1 986772
23	1 361728	48	1 681241	73	1 863323	98	1 991226
24	1 380211	49	1 690196	74	1 869232	99	1 995635
25	1 397940	50	1 698970	75	1 875061	100	2 000000

TABLE II.—LOGARITHMS OF LEADING NUMBERS WITHOUT INDICES.

N.	0.	1.	2.	3.	4.	5.	6.	7.	8.	9.
100	000000	000434	000868	001301	001734	002166	002598	003029	003461	003891
101	004321	004750	005181	005609	006038	006466	006894	007321	007748	008174
102	008600	009026	009451	009876	010300	010724	011147	011570	011993	012415
103	012837	013259	013680	014100	014521	014940	015360	015779	016197	016616
104	017033	017451	017868	018284	018700	019116	019532	019947	020361	020775
105	021189	021603	022016	022428	022841	023252	023664	024075	024486	024896
106	025306	025715	026125	026533	026942	027350	027757	028164	028571	028978
107	029384	029789	030195	030600	031004	031408	031812	032216	032619	033021
108	033424	033826	034227	034628	035029	035430	035830	036230	036629	037028
109	037426	037825	038223	038620	039017	039414	039811	040207	040602	040998

In table I, the logarithms are given, with indices, in columns adjacent to the columns of numbers.

In table II, each figure in the row at the top may be annexed to any number in the left-hand column; the logarithm of any number thus formed, will be found at the right of the number in the column, and beneath the figure at the top. The proper index may be supplied in any case, according to the theory of logarithms. Thus, to obtain the logarithm of 1023 by this table, we find 102 in the left-hand column, and 3 in the top row; and opposite the former, and under the latter, we find 009876, the decimal part of the logarithm. Hence, log. 1023 = 3.009876.

In like manner, we find

$$\log.\ 104.2 = 2.017868, \qquad \log.\ .1078 = -1.032619.$$

CASE I.

416. To find the logarithms of numbers when their factors are in the tables.

RULE.—*Take out from the tables the logarithms of the factors, and find their sum; the result will be the logarithm required.*

EXAMPLES FOR PRACTICE.

1. Required the logarithm of 533.5.

Observe that

$$\begin{aligned} 533.5 &= 106.7 \times 5; \\ \log.\ 106.7 &= 2.028164 \\ \log.\ 5 \quad &= \ \ .698970 \\ \hline & \quad 2.727134,\ \textit{Ans.} \end{aligned}$$

2. Find the logarithm of 520.	*Ans.* 2.716003.
3. Find the logarithm of 146.	*Ans.* 2.164353.
4. Find the logarithm of 1450.	*Ans.* 3.161368.
5. Find the logarithm of 1.59.	*Ans.* .201397.
6. Find the logarithm of 2034.	*Ans.* 3.308351.
7. Find the logarithm of 76.37.	*Ans.* 1.882923.
8. Find the logarithm of .0201.	*Ans.* $-$ 2.303196.
9. Find the logarithm of .3822.	*Ans.* $-$1.582290.
10. Find the logarithm of 16995.	*Ans.* 4.230321.

CASE II.

417. To find the logarithms of numbers intermediate between the numbers in the table.

Since the logarithms in any table form a regular series, we may *interpolate* for intermediate logarithms, by the usual formula,

$$T_{1+n} = a + nd_1 + \frac{n(n-1)}{2} d_2 + \dots.$$

If the logarithm of the given number is intermediate between the logarithms of table I, it will be necessary to take account of the *first* and *second* differences. But we may always employ table II, where the logarithms increase so slowly that two terms of the formula will give the result accurately.

The first four figures of a number, counting from the left, will be called the four *superior figures;* and the others, the *inferior figures.* To apply the formula, a will represent that logarithm of the table which is next less than the required logarithm, and n will denote the inferior figures of the number, *regarded as a decimal.*

Hence the following

RULE.—*Take out the logarithm of the four superior figures of the given number; multiply the difference between this logarithm and the next greater in the table, by the inferior places of the number, considered as a decimal; add this product to the former result, and the sum will be the logarithm required.*

EXAMPLES FOR PRACTICE.

1. Required the logarithm of 1.07632.

This number is found between 1.076 and 1.077; hence

$$\text{log. } 1.077 - \text{log. } 1.076 = 404 = d_1.$$

And putting $n = .32$, we have

$$\begin{aligned} a = \text{log. } 1.076 &= .031812 \\ nd_1 = 404 \times .32 &= \underline{129} \\ & .031941, \textit{ Ans.} \end{aligned}$$

2. Required the logarithm of 3579.

In order to make use of table II, we proceed thus:

$$3579 \div 35 = 102.25714+.$$

$$\text{log. } 102.3 - \text{log. } 102.2 = 425; \qquad n = .5714.$$

$$\begin{aligned} \text{log. } 102.2 &= 2.009451 \\ 425 \times .5714 &= \underline{243} \\ & 2.009694 = \text{log. } 102.25714 \\ \text{log. } 35 &= \underline{1.544068} \\ & 3.553762, \textit{ Ans.} \end{aligned}$$

NOTE.—It is obvious that if we divide any number by its first two figures, we may obtain the logarithm of the quotient by means of table II; then we may add the logarithm of the divisor, found by table I, to obtain the required logarithm.

3. Find the logarithm of 10724. *Ans.* 4.030357.
4. Find the logarithm of 10.8539. *Ans.* 1.035586.
5. Find the logarithm of 1021.56. *Ans.* 3.009264.
6. Find the logarithm of 568.53. *Ans.* 2.754753.
7. Find the logarithm of 3244. *Ans.* 3.511081.
8. Find the logarithm of 365.25638. *Ans.* 2.562598.
9. Find the logarithm of 132.57. *Ans.* 2.122445.
10. Find the logarithm of 567521. *Ans.* 5.753982.
11. Find the logarithm of 258.7. *Ans.* 2.412796.
12. Find the logarithm of 1.296. *Ans.* .112605.
13. Find the logarithm of 5784. *Ans.* 3.762228.

EXPONENTIAL EQUATIONS.

418. We will now illustrate the application of logarithms to the solution of exponential equations.

1. Given $2^x = 10$ to find the value of x.

Suppose the logarithms of both members of the equation to be taken. We shall have, by **404,** 5,

$$x \log. 2 = \log. 10\,;$$

or, $$x = \frac{\log. 10}{\log. 2} = \frac{1}{.301030} = 3.3219+, \textit{ Ans.}$$

2. Given $5^{\frac{2}{x}} = \frac{3}{7}$ to find the value of x.

Raising both members of the given equation to the power denoted by x, we have

$$25 = \frac{3^x}{7^x}.$$

Taking the logarithms of both members,

$$\log. 25 = x \log. 3 - x \log. 7\,; \qquad \text{whence,}$$

$$x = \frac{\log. 25}{\log. 3 - \log. 7} = \frac{1.397940}{.477121 - .845098} = -3.79899+, \textit{ Ans.}$$

3. Given $ra^x = b^2c$ to find the value of x.

Taking the logarithms of both members of the equation, we have, by **404,** 3 and 5,

$$\log. r + x \log. a = 2 \log. b + \log. c\,;$$

whence, $$x = \frac{2 \log. b + \log. c - \log. r}{\log. a}.$$

EXAMPLES FOR PRACTICE.

1. Given $7^x = 8$ to find the value of x. *Ans.* $x = 1.06862$.

2. Given $5^{\frac{2}{x}} = 30$ to find the value of x. *Ans.* $x = .94640$.

3. Given $a^x = b^2c^3$ to find the value of x.

$$\textit{Ans. } x = \frac{2 \log. b + 3 \log. c}{\log. a}.$$

4. Given $\dfrac{ab^x - c}{d} = m$ to find the value of x.

$$Ans.\ \ x = \frac{\log.(md + c) - \log.a}{\log.b}.$$

5. Given $ma^{\frac{1}{x}} = b$ to find the value of x.

$$Ans.\ \ x = \frac{\log.a}{\log.b - \log.m}.$$

6. Given $a^x + b^y = 2c$ and $a^x - b^y = 2d$ to find x and y.

$$Ans.\ \ x = \frac{\log.(c + d)}{\log.a},\ y = \frac{\log.(c - d)}{\log.b}.$$

7. Given $729^{\frac{1}{x}} = 3$ to find the value of x. $\quad$ *Ans.* $x = 6$.

8. Given $216^{\frac{3}{x}} = 12$ to find the value of x.

$$Ans.\ \ x = \frac{9\log.6}{\log.12}.$$

9. Given $516^{\frac{3}{x}} = 12$ to find the value of x.

$$Ans.\ \ x = \frac{3\log.43}{\log.12} + 3.$$

10. Given $6^x = \dfrac{24^6(17)^{\frac{1}{3}}}{71}$ to find the value of x.

$$Ans.\ \ x = \frac{18\log.24 + \log.17 - 3\log.71}{3\log.6}.$$

SECTION VIII.

PROPERTIES OF EQUATIONS.

419. Let us assume the equation,

$$x^m + Ax^{m-1} + Bx^{m-2} + \ldots. + Tx + U = 0 \quad . \quad . \quad . \quad (1),$$

in which m, the exponent of the degree, is a positive whole number. An equation not given in this form may be readily reduced to it, by transposing all the terms to the first member, arranging them according to the descending powers of the unknown quantity, and dividing through by the coefficient of the first term.

In this equation the coefficients, A, B, C, etc., may denote any quantities whatever; that is, they may be positive or negative, entire or fractional, rational or irrational, real or imaginary.

The term U may be regarded as the coefficient of x^0, and is called the *absolute* term of the equation.

420. If the equation contains all the entire powers of x, from the m^{th} down to the zero power, it is said to be complete; if some of the intermediate powers of x are wanting, it is said to be incomplete. An incomplete equation may be made to take the form of a complete equation, by writing the absent powers of x with ± 0 for their coefficient.

421. It has been shown (**305**) that any expression of the second degree containing but one unknown quantity, may be resolved into two binomial factors of the first degree with respect to the unknown quantity,—the first term in each factor being this quantity, and the second term one of the roots (with its sign changed) of the equation which results from placing the expression equal to zero. We therefore conclude that every expression of the second degree may be regarded as the product of two binomial factors of the first degree.

So likewise the product of three binomial factors of the first degree with respect to any unknown quantity, will be an expres-

sion of the third degree, and we readily see that by varying the values of the second terms of the factors, corresponding changes are produced in the product. Thus,

$$(x-2)(x+3)(x-5) = x^3 - 4x^2 - 11x + 30,$$
$$\left(x-2+\sqrt{-3}\right)\left(x-2-\sqrt{-3}\right)(x+\tfrac{7}{4}) = x^3 - \tfrac{9}{4}x^2 + \tfrac{49}{4},$$
$$\left(x+1-\sqrt{-3}\right)\left(x+1+\sqrt{-3}\right)(x-2) = x^3 - 8.$$

From these and other examples, which may be increased at pleasure, it is inferred that any expression of the third degree in respect to x, would result from the multiplication of some three factors of the first degree in respect to x. And in general, any expression of the m^{th} degree with respect to its unknown quantity, may be regarded as the result of the multiplication of m binomial factors of the first degree with respect to that unknown quantity.

422. If then we have any equation formed by placing a polynomial containing the unknown quantity, x, equal to zero, and we discover the binomial factor $x - a$ in the first member, it is evident that a is a root of the equation; for, when substituted for x, it reduces the first member to zero.

If we can succeed, therefore, in discovering the binomial factors of the first degree, of the first member of any equation, the roots of the equation will be the values of x obtained by placing each of these factors, successively, equal to zero.

This reverse process of resolving the first member of an equation into its binomial factors of the first degree, is one the difficulty of which increases rapidly with the degree of the equation; and algebraists have as yet discovered no general method for effecting this resolution for those of a higher degree than the fourth. By special processes, however, the roots of *numerical* equations may be found exactly, when commensurable, and to any degree of approximation when not commensurable.

423. In order to discover the law which governs the product of any number of binomial factors, such as $x + a$, $x + b$, $x + c$, etc., having the first term the same in all, and the second terms different, let us first obtain the product of several of these factors by actual multiplication; thus,

$$\begin{array}{l|l}
x + a & \\
x + b & \\
\hline
x^2 + a & x + ab \\
\quad + b &
\end{array} \left.\vphantom{\begin{array}{l}1\\1\end{array}}\right\} = (x+a)(x+b)$$

$$\begin{array}{l|l|l}
x + c & & \\
\hline
x^3 + a & x^2 + ab & x + abc \\
\quad + b & \quad + ac & \\
\quad + c & \quad + bc &
\end{array} \left.\vphantom{\begin{array}{l}1\\1\\1\end{array}}\right\} = (x+a)(x+b)(x+c)$$

$$\begin{array}{l|l|l|l}
x + d & & & \\
\hline
x^4 + a & x^3 + ab & x^2 + abc & x + abcd \\
\quad + b & \quad + ac & \quad + abd & \\
\quad + c & \quad + bc & \quad + acd & \\
\quad + d & \quad + ad & \quad + bcd & \\
 & \quad + bd & & \\
 & \quad + cd & &
\end{array} \left.\vphantom{\begin{array}{l}1\\1\\1\\1\\1\\1\end{array}}\right\} = (x+a)(x+b)(x+c)(x+d).$$

From an examination of these several products we arrive at the following conclusions:

1. The exponent of the leading letter, x, in the first term is equal to the number of binomial factors used, and this exponent decreases by 1, from term to term, towards the right, until we come to the last term, in which it is 0.

2. The coefficient of the first term is 1; that of the second, the sum of the second terms of the binomial factors; that of the third, the sum of all the different products formed by multiplying, two and two, the second terms of the binomial factors; that of the fourth, the sum of all the different products formed by multiplying, three and three, the second terms of the binomial factors; the last, or absolute term, is the continued product of the second terms of the binomial factors.

It might be inferred from what has been now shown, that however great the number of binomial factors employed, the coefficient of that term of the arranged product which has n terms before it, would be the sum of all the different products that can be formed by multiplying the second terms of the binomial factors in sets of n and n.

Assuming that the above law holds true for a number m, of binomial factors, if it can be proved that it still governs the

product when an additional factor is introduced, it will be established in all its generality. Let us suppose then, that in the product,

$$x^m + Ax^{m-1} + Bx^{m-2} + \ldots . \, Mx^{m-n+1} + Nx^{m-n} + \ldots . + Tx + U,$$

of the m binomial factors $x + a$, $x + b$, ..., $x + p$, the law of formation is the same as that found by the actual multiplication of several factors.

Introducing the factor $x + q$, we have

$$\begin{array}{l}
x^m + Ax^{m-1} + Bx^{m-2} + \ldots . \, Mx^{m-n+1} + Nx^{m-n} + \ldots . \, Tx + U \\
\underline{x + q \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad}
\end{array}$$

$$\begin{array}{r|r|r|r|r|l}
x^{m+1} + A & x^m + B & x^{m-1} + \ldots . + N & x^{m-n+1} + \ldots . + U & x & \\
+q & +Aq & +Mq & +Tq & & + Uq.
\end{array}$$

It is at once seen that, in this new product, the law in respect to the exponents is unbroken. As to the coefficients, that of the first term is still 1; that of the second term is $A + q$; and since A is the sum of the second terms of the m factors in the assumed product, $A + q$ is the sum of the second terms of the $m + 1$ binomials. The coefficient of the third term is $B + Aq$. Now B is, by hypothesis, the sum of the different products of the second terms, of the m binomial factors, taken two and two in a set, and Aq is all of the additional products to which the introduction of the factor $x + q$ can give rise; hence $B + Aq$ is the sum of all the products, taken two and two, of the second terms of the $m + 1$ binomials. And the coefficient of the general term, that is the coefficient of the term having n terms before it, is $N + Mq$; but N is the sum of all the products of the second terms, taken n and n, of the binomial factors which enter the assumed product; and because M is the sum of all the products of these second terms, taken $n - 1$ and $n - 1$, Mq is the sum of all the additional products, taken n and n, which can result from the introduction of the factor $x + q$.

Now we have proved, by actual multiplication, that the law of the product, assumed to be true for m binomial factors, is true for four factors; hence by what has just been demonstrated, it is true for five factors; and being true for five, it must be true for six, and so on. Therefore the law is general.

424. The composition of the coefficients of an equation in terms of its roots.

Let us take any number, m, of binomial factors, as $x-a$, $x-b$, $x-c$, ... $x-p$, $x-q$, in which a, b, c, etc., may represent any quantities whatever. Now it has been shown (**423**) that the continued product of these factors, arranged according to the descending powers of x, will be of the form

$$x^m + Ax^{m-1} + Bx^{m-2} + Cx^{m-3} + \ldots . Sx^2 + Tx + U,$$

in which

$$\begin{aligned} A &= -a - b - c - \ldots . - p - q, \\ B &= + ab + ac + bc + \ldots . + ap + aq, \\ C &= - abc - bcd - acd - \ldots . - abp - abq, \\ &\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ S &= \pm\, abcd \ldots pq_{m-2} \pm bcde \ldots . pq_{m-2} \pm \text{etc.} \\ T &= \mp\, abcde \ldots . pq_{m-1} \mp bcdef \ldots . pq_{m-1} \mp \text{etc.} \\ U &= \pm\, abcd \ldots . pq_m, \end{aligned}$$

the subscript expressions $m-2$, $m-1$, m, denoting the number of literal factors which enter each term.

We thus have the identical equation,

$$\left.\begin{matrix}(x-a)(x-b)(x-c) \ldots . \\ (x-p)(x-q)\end{matrix}\right\} = \left\{\begin{matrix} x^m + Ax^{m-1} + Bx^{m-2} + \ldots . \\ + Sx^2 + Tx + U \end{matrix}\right\} \ldots (1),$$

and placing the second member of this equal to zero we have

$$x^m + Ax^{m-1} + Bx^{m-2} + \ldots . Sx^2 + Tx + U = 0 \ldots (2),$$

an equation of which a, b, c, ... p, q are the roots, since these values substituted in succession for x in the first member of (1) will cause this first member, and consequently the second member, to vanish. The relations between the coefficients A, B, C, etc., and the roots of (2), may be expressed as follows:

1. *The coefficient of the second term is equal to the algebraic sum of all the roots, with the signs changed.*

2. *The coefficient of the third term is equal to the algebraic sum of all the different products formed by multiplying the roots, two and two.*

3. *The coefficient of the fourth term is equal to the algebraic sum of all the different products formed by multiplying the roots, with their signs changed, three and three.*

4. And in general: *The coefficient of the term having* n *terms before it, is equal to the algebraic sum of all the different products formed by multiplying the roots, with their signs changed if* n *is odd,* n *and* n. Hence,

5. *The absolute term is the continued product of all the roots, with their signs changed when the number denoting the degree of the equation is odd.*

This principle will enable us to construct an equation, the roots of which are given, and the composition of (1) shows that (2) thus constructed can have no other than the assumed roots; for there is no value of x differing from one of these roots which can cause the first member of (1) to disappear.

From this we might conclude that every equation involving but one unknown quantity, has as many roots as there are units in the exponent of its degree, and can have no more.

425. Admitting that every equation containing but one unknown quantity has at least one root, real or imaginary, it may be demonstrated that the first member of every equation of the m^{th} degree, the second member being zero, may be regarded as the continued product of m binomial factors of the first degree with respect to the unknown quantity. We will first prove that,

If a *is a root of an equation of the form*

$$x^m + Ax^{m-1} + Bx^{m-2} + \dots . Tx + U = 0 \quad . \quad . \quad . \quad (1),$$

its first member can be exactly divided by x — a.

For if we apply the rule for division, we shall finally arrive at a remainder which will not contain x; since for each quotient term obtained, the new dividend is at least one degree lower than that which precedes.

Calling the entire quotient Q and the remainder R, we shall have

$$x^m + Ax^{m-1} + Bx^{m-2} + \dots . Tx + U = Q(x-a) + R \quad . \quad . \quad . \quad (2),$$

an identical equation. The substitution of a for x causes the first member, and also the first term in the second member of this equation, to vanish. Hence, $R = 0$. But by hypothesis R does not contain x; it is therefore equal to zero whatever value be attributed to x, and the division is exact.

426. The converse of the last principle is also true; that is,

If the first member of the equation,

$$x^m + Ax^{m-1} + Bx^{m-2} + \dots. Tx + U = 0,$$

can be exactly divided by x — a, *then* a *is a root of the equation.*

For, suppose the division performed, and that the quotient is Q; then we shall have the identical equation,

$$x^m + Ax^{m-1} + Bx^{m-2} + \dots. Tx + U = Q(x - a).$$

But $x = a$ causes the second member of this equation to vanish; it will therefore cause the first member to vanish, and consequently satisfy the given equation.

427. Every equation containing but one unknown quantity has a number of roots denoted by the exponent of its degree, and no more.

Resuming the equation,

$$x^m + Ax^{m-1} + Bx^{m-2} + \dots. Tx + U = 0,$$

and admitting that it has one root, a, $x - a$ must be a factor of its first member (**425**). The quotient which arises from the division of the polynomial,

$$x^m + Ax^{m-1} + \dots. Tx + U,$$

by $x - a$, will be of the form

$$x^{m-1} + A'x^{m-2} + \dots. T'x + U';$$

we shall therefore have the identical equation,

$$\left.\begin{matrix} x^m + Ax^{m-1} + Bx^{m-2} \\ + \dots. Tx + U \end{matrix}\right\} = (x-a)(x^{m-1} + A'x^{m-2} + \dots. T'x + U').$$

Now the second member of this equation will vanish for any value of x which reduces the second factor to zero.

If then the assumed root of the equation,

$$x^{m-1} + A'x^{m-2} + \dots. T'x + U' = 0,$$

be denoted by b, we shall have

$$\left.\begin{matrix} x^{m-1} + A'x^{m-2} + \\ \dots. T'x + U' \end{matrix}\right\} = (x-b)(x^{m-2} + A''x^{m-3} + \dots. T''x + U'').$$

A third equation may be formed in the same way, and then a fourth, and so on, until the $(m-1)^{th}$ equation is finally reached, in which the second factor in the second member is of the first degree with respect to x.

Taking this last equation, and substituting for its first member the second, in the next preceding equation, and thus continuing the process of substitution until the first equation of the series is arrived at, the result will be the following identical equation:

$$\left.\begin{matrix} x^m + Ax^{m-1} + Bx^{m-2} + \\ \dots . Tx + U \end{matrix}\right\} = \left\{\begin{matrix} (x-a)(x-b)(x-c)\dots . \\ (x-p)(x-q). \end{matrix}\right.$$

The second member of this equation vanishes for any one of the m values,

$$x = a,\ x = b,\ x = c,\ \dots . x = p,\ x = q,$$

and consequently these values are severally roots of the equation,

$$x^m + Ax^{m-1} + Bx^{m-2}\dots . Tx + U = 0.$$

Moreover, no value of x that differs from some one of these values, can satisfy the equation; for no such value will cause any one of the factors in the second member of the identical equation to be zero, a condition requisite to make the product zero. The equation therefore has m roots and no more.

428. From the foregoing principles we conclude,

1. That in an equation in which the second term does not appear,—that is, the term containing the next to the highest power of the unknown quantity,—the algebraic sum of the roots is 0.

2. If an equation has no absolute term, at least one of its roots is 0.

3. The absolute term being the continued product of all the roots of an equation, it must be exactly divisible by each of them.

4. An equation may be constructed, which shall have any assumed roots.

5. The degree of an equation may be reduced by 1 for each of its known roots.

EXAMPLES.

1. What is the equation having $+2$, -3 for its roots?
Ans. $x^2 + x - 6 = 0$.

2. What is the equation having the roots $+1$, -2, -4?
Ans. $x^3 + 5x^2 + 2x - 8 = 0$.

3. What is the equation having for its roots $+3$, -2, -1, $+5$? *Ans.* $x^4 - 5x^3 - 7x^2 + 29x + 30 = 0$.

4. What is the equation of which the roots are $1+\sqrt{-5}$, $1-\sqrt{-5}$, $+\sqrt{5}$, $-\sqrt{5}$? *Ans.* $x^4-2x^3+x^2+10x-30=0$.

5. What is the equation of which the roots are -1, -2, $+3$, $2+\sqrt{-3}$, $2-\sqrt{-3}$? *Ans.* $x^5-4x^4+22x^2-25x-42=0$.

6. One root of the equation,

$$x^3-5x^2+13x-21=0,$$

is $+3$; what is the reduced equation? *Ans.* $x^2-2x+7=0$.

7. One root of the equation,

$$x^4+2x^3-34x^2+12x+35=0,$$

is -7; what is the depressed equation?

Ans. $x^3-5x^2+x+5=0$.

8. Two of the roots of the equation,

$$x^4-3x^3-4x^2+30x-36=0,$$

are $+2$, -3; what is the depressed equation, and what are its roots?

Ans. The depressed equation is $x^2-4x+6=0$; and its roots are $2+\sqrt{-2}$, $2-\sqrt{-2}$.

429. Any equation having fractional coefficients can be transformed into another in which the coefficients are entire, that of the first term being unity.

If the coefficient of the first term of the given equation is not unity, make it so by dividing through by this coefficient. Then the equation will be of the form

$$x^m+Ax^{m-1}+Bx^{m-2}+\ldots\ldots Tx+U=0,$$

in which it is supposed that some or all the coefficients, A, B, etc., are fractional.

Assume $x=\frac{y}{a}$, a being entirely arbitrary, and substitute this value of x in the equation; it then becomes

$$\frac{y^m}{a^m}+A\frac{y^{m-1}}{a^{m-1}}+B\frac{y^{m-2}}{a^{m-2}}+\ldots\ldots T\frac{y}{a}+U=0.$$

Whence, by multiplying through by a^m,

$$y^m+Aay^{m-1}+Ba^2y^{m-2}+\ldots\ldots Ta^{m-1}y+Ua^m=0.$$

Now since a is arbitrary, its value may be so selected that it and

its powers will contain the denominators of the fractional coefficients of the original equations. We present the following examples for illustration.

1. Transform the equation,

$$x^3 + \frac{ax^2}{m} + \frac{bx}{n} + \frac{c}{p} = 0,$$

into another which shall have no fractional coefficients, and which shall have unity for its first coefficient.

Make $x = \frac{y}{mnp}$; substituting this value of x, the equation becomes

$$\frac{y^3}{m^3n^3p^3} + \frac{ay^2}{m^3n^2p^2} + \frac{by}{mn^2p} + \frac{c}{p} = 0.$$

Multiplying every term of this by $m^3n^3p^3$,

$$y^3 + anpy^2 + bm^2np^2y + cm^3n^3p^2 = 0.$$

When the denominators of the coefficients have common factors, we may make x equal to y divided by the least common multiple of the denominators.

2. Transform the equation, $x^3 + \frac{ax^2}{pm} + \frac{bx}{m} + \frac{c}{p} = 0$, into another which shall have no fractional coefficients, and that of the first term be unity.

To effect this it is sufficient to put $x = \frac{y}{pm}$. With this value of x the equation becomes

$$\frac{y^3}{p^3m^3} + \frac{ay^2}{p^3m^3} + \frac{by}{pm^2} + \frac{c}{p} = 0.$$

Multiplying every term by p^3m^3,

$$y^3 + ay^2 + bp^2my + cp^2m^3 = 0,$$

the transformed equation required.

3. Transform the equation $x^4 + \frac{5x^3}{6} + \frac{3x^2}{4} + \frac{7x}{24} + \frac{1}{12} = 0$ into another having no fractional coefficients.

Ans. $y^4 + 20y^3 + 18 \cdot 24y^2 + 7\,(24)^2 y + 2\,(24)^3 = 0.$

In transforming an equation having fractional, into another with entire coefficients, in terms of another unknown quantity, it is important to have the transformed equation in the lowest possible terms. The least common multiple of the denominators will not necessarily be the least value of a that will give the required equation. If, in each case, the denominators be resolved into their prime factors, it will be easy to decide upon the powers of these factors to be taken as the factors of a.

The following illustration will render further explanation unnecessary.

4. Transform the equation,

$$x^3 - \frac{3}{35}x^2 + \frac{13}{2450}x - \frac{17}{68600} = 0,$$

into another of the same form with the smallest possible entire coefficients.

Writing y for x and multiplying the second, third and fourth terms, by a, a^2, a^3, respectively, we have

$$y^3 - \frac{3}{35}ay^2 + \frac{13}{2450}a^2x - \frac{17}{68600}a^3 = 0.$$

The denominators, resolved into their prime factors, are

$$7 \cdot 5, \quad 7^2 \cdot 5^2 \cdot 2, \quad 7^3 \cdot 5^2 \cdot 2^3;$$

and assuming $a = 7 \cdot 5 \cdot 2$, the equation may be written

$$y^3 - \frac{3 \cdot 7 \cdot 5 \cdot 2}{7 \cdot 5}y^2 + \frac{13 \cdot 7^2 \cdot 5^2 \cdot 2^2}{7^2 \cdot 5^2 \cdot 2}y - \frac{17 \cdot 7^3 \cdot 5^3 \cdot 2^3}{7^3 \cdot 5^2 \cdot 2^3} = 0,$$

which reduces to

$$y^3 - 6y^2 + 26y - 85 = 0.$$

In this example, the least common multiple of the denominators is $7^3 \cdot 5^2 \cdot 2^3$; and had this value been taken for a, instead of $7 \cdot 5 \cdot 2$, the coefficients of the transformed equation would have been much larger than they are, as found above.

When a root of the transformed equation is known, the corresponding root of the original equation will be given by the relation

$$x = \frac{y}{a}.$$

COMMENSURABLE ROOTS.

430. A number is commensurable with unity when it can be expressed by an exact number of units or parts of a unit; a number which cannot be so expressed is incommensurable with unity.

431. Every equation having unity for the coefficient of the first term, and for all the other coefficients, whole numbers, can have only whole numbers for its commensurable roots.

This being one of the most important principles in the theory of equations, its enunciation should be clearly understood. Such equations may have *other roots* than whole numbers; but its roots cannot be among the definite and irreducible fractions, such as $\frac{3}{5}$, $\frac{7}{8}$, $\frac{11}{4}$, etc. Its other roots must be among the incommensurable quantities, such as $\sqrt{2}$, $(3)^{\frac{1}{3}}$, etc.; *i. e.*, surds, indeterminate decimals, or imaginary quantities.

To prove the proposition, let us suppose $\frac{a}{b}$, a commensurable but irreducible fraction, to be a root of the equation,

$$x^m + Ax^{m-1} + Bx^{m-2} \dots \dots Tx + U = 0.$$

A, B, etc., being whole numbers.

Substituting this supposed value of x, we have

$$\frac{a^m}{b^m} + A\frac{a^{m-1}}{b^{m-1}} + B\frac{a^{m-2}}{b^{m-2}} \dots \dots T\frac{a}{b} + U = 0.$$

Transpose all the terms but the first, and multiply by b^{m-1}, and we have

$$\frac{a^m}{b} = -\left(Aa^{m-1} + Ba^{m-2}b \dots \dots Tab^{m-2} + Ub^{m-1}\right).$$

Now, as a and b are prime to each other, b cannot divide a, and it cannot possibly divide any power of a; because $\frac{a}{b}$ being irreducible, $\frac{a}{b} \times a$ is also irreducible, as the multiplier a will not be measured by the divisor b; therefore $\frac{a^2}{b}$ cannot be expressed in whole numbers. Continuing the same mode of reasoning, $\frac{a^m}{b}$

cannot express a whole number, but every term in the other member of the equation expresses a whole number.

Hence, the supposition that the irreducible fraction $\frac{a}{b}$ is a root of the equation, leads to this absurdity, that a series of whole numbers is equal to an irreducible fraction.

Therefore, we conclude that any equation corresponding to these conditions cannot have a definite *commensurable* fraction among its roots.

432. It has been shown (**429**) that an equation having fractional coefficients, that of the first term being unity, can be changed into another of the same form, with entire coefficients. The expression *entire* must there be understood in its algebraic sense; that is, the new coefficients being entire merely in algebraic form, may be irrational or imaginary. In the preceding article it is proved that if these coefficients are whole numbers, all the commensurable roots of the equation are also whole numbers; moreover, these roots must be found among the divisors of the absolute term (**428**). If the divisors of the absolute term are few and obvious, those answering to the roots may be found by trial substitutions; but in most cases the labor will be abridged by the rule suggested by the following investigation:

Suppose a to be a commensurable root of the equation,

$$x^m + Ax^{m-1} + \dots + Rx^3 + Sx^2 + Tx + U = 0.$$

Writing a for x, transposing all the terms except the last to the second member, and dividing through by a, we have

$$\frac{U}{a} = -a^{m-1} - Aa^{m-2} - \dots - Ra^2 - Sa - T.$$

But, since a is a root of the equation, $\frac{U}{a}$ is an entire number; transpose $-T$ to the first member of the last equation, make $\frac{U}{a} + T = N_1$, and divide both members of the resulting equation by a; it then becomes

$$\frac{N_1}{a} = -a^{m-2} - Aa^{m-3} - \dots - Ra - S.$$

The second member of this equation is a whole number; the

first member is therefore entire; and if $-S$ be transposed to this member, and $\frac{N_1}{a}+S$ be denoted by N_2, we shall have, after again dividing through by a, the equation,

$$\frac{N_2}{a}=-a^{m-3}-Aa^{m-4}-\ldots\ldots-R,$$

of which the second, and therefore the first member, is a whole number.

By continuing this process of transposition and division, we shall finally arrive at the equations,

$$\frac{N_{m-2}}{a}=-a-A,$$

$$\frac{N_{m-1}}{a}=-1, \text{ or } \frac{N_{m-1}}{a}+1=0.$$

Every whole number which is a root of the proposed equation will satisfy all of the above conditions, from the first down to that expressed by the equation,

$$\frac{N_{m-1}}{a}=-1,$$

by which the root will be recognized.

All of the commensurable roots of an equation of the assumed form may then be found by the following

RULE.—I. *Write all of the exact divisors of the absolute term in a line, and beneath them write their respective quotients.*

II. *Add to these quotients, severally, the coefficient of the next to the last term, with its proper sign.*

III. *Divide such of these sums by the divisors to which they correspond as will give exact quotients, neglecting others.*

IV. *Add to these quotients the coefficient of the third term from the last, with its proper sign, and divide again as before, and so on, until the coefficient of the second term has been added to the preceding quotients, and these last in turn are divided by their respective divisors. Those divisors which correspond to the final quotients, minus* 1, *are roots of the equation.*

NOTE.—Absent powers of the unknown quantity must be introduced with ± 0 for their coefficient.

EXAMPLES.

1. Required the commensurable roots (if any) of the equation,

$$x^4 + 4x^3 - 3x^2 + 8x - 10 = 0.$$

OPERATION.

Divisors,	10,	5,	2,	1,	-1,	-2,	-5,	-10.
	-10							
Quotients,	-1,	-2,	-5,	-10,	10,	5,	2,	1.
Add	8							
	7,	6,	3,	-2,	18,	13,	10,	9.
2d quotients,				-2,	-18,		-2.	
Add				-3,				
				-5,	-21,		-5.	
3d quotients,				-5,	21,		1.	
Add				4				
				-1,	25,		5.	
4th quotients,				-1,	-25,		-1.	

Thus, there are two final quotients equal to -1; and the corresponding divisors are 1 and -5. Hence, the given equation has two commensurable roots, 1 and -5.

If we divide the given equation by $(x - 1)(x + 5)$, or $x^2 + 4x - 5$, the quotient will be $x^2 - 2$. Hence,

$$x^2 - 2 = 0, \qquad x = \pm\sqrt{2};$$

and the four roots are $1, -5, +\sqrt{2}, -\sqrt{2}$.

2. Required the commensurable roots of the equation, $x^3 - 6x^2 + 11x - 6 = 0$. *Ans.* 1, 2, 3.

3. Required the commensurable roots of the equation, $x^4 + 4x^3 - x^2 - 16x - 12 = 0$. *Ans.* $2, -1, -2, -3$.

4. Required the commensurable roots of the equation, $x^4 - 6x^2 - 16x + 21 = 0$. *Ans.* 3 and 1.

NOTE.—Supplying the absent term, the equation will be $x^4 \pm 0x^3 - 6x^2 - 16x + 21 = 0$. In the operation, go through the form of adding 0.

5. It is required to find all the roots of the equation, $x^4 - 6x^3 + 5x^2 + 2x - 10 = 0$. *Ans.* $-1, +5, 1 + \sqrt{-1}, 1 - \sqrt{-1}$.

DERIVED POLYNOMIALS.

433. The first member of an equation involving but one unknown quantity, to which all the terms have been transposed, is a polynomial in the most general sense of the term, and may be operated on as an algebraic quantity, without reference to the equation and to the particular values of the unknown quantity which will reduce the polynomial to zero, or satisfy the equation.

If we take the polynomial,

$$Ax^m + Bx^{m-1} + Cx^{m-2} + \dots + Sx^2 + Tx + U,$$

and multiply each term by the exponent of x in that term, then diminish this exponent by unity, and form the algebraic sum of the results, we shall have

$$mAx^{m-1} + (m-1)Bx^{m-2} + (m-2)Cx^{m-3} + \dots + 2Sx + T.$$

Constructing a third polynomial from this, in the same way that this was derived from the first, we have $m(m-1)Ax^{m-2} +$

$$(m-1)(m-2)Bx^{m-3} + (m-2)(m-3)Cx^{m-4} + \dots + 2S.$$

A fourth may be formed from the third according to the same law; and so on, until we arrive at an expression which will be independent of x, because the degree, with respect to x, of any polynomial thus formed, is one less than of that which immediately precedes it.

Denoting the given polynomial by X, the second by X_1, the third by X_2, etc., then

X_1 is the first derived polynomial of				X,
X_2	"	"	"	X_1,
X_3	"	"	"	X_2, etc.

And X_1, X_2, X_3 are the *successive derived polynomials* of X, and are called *first, second, third,* etc., derived polynomials.

Preserving the above notation, we have for the successive derived polynomials, the following

RULE.—*To form* X_1, *multiply every term of* X *by the exponent of small* x *in the term, then diminish this exponent by unity and take the algebraic sum of the results.* X_2 *is derived from* X_1 *in the same way that* X_1 *is derived from* X; *and so on.*

What are the successive derived polynomials of

$$3x^5 + 5x^4 - 9x^3 + 7x^2 - 8x + 5?$$

$$Ans. \begin{cases} \text{1st,} & 5\cdot 3x^4 + 4\cdot 5x^3 - 3\cdot 9x^2 + 2\cdot 7x - 8; \\ \text{2d,} & 4\cdot 5\cdot 3x^3 + 3\cdot 4\cdot 5x^2 - 2\cdot 3\cdot 9x + 2\cdot 7; \\ \text{3d,} & 3\cdot 4\cdot 5\cdot 3x^2 + 2\cdot 3\cdot 4\cdot 5x - 2\cdot 3\cdot 9; \\ \text{4th,} & 2\cdot 3\cdot 4\cdot 5\cdot 3x + 2\cdot 3\cdot 4\cdot 5; \\ \text{5th,} & 2\cdot 3\cdot 4\cdot 5\cdot 3. \end{cases}$$

COMPOSITION OF DERIVED POLYNOMIALS.

434. Let us take the polynomial,

$$x^m + Ax^{m-1} + Bx^{m-2} + \ldots + Tx + U = X;$$

and suppose that its binomial factors of the first degree with respect to x are

$$x - a,\ x - b,\ x - c,\ \ldots,\ x - m,\ x - n.$$

We shall then have the identical equation,

$$x^m + Ax^{m-1} + \ldots + Tx + U = (x-a)(x-b)\ldots(x-m)(x-n),$$

which will subsist as a true equation, whatever quantity be substituted for x in its two members. Replace x by $y + x$; then

$$(y+x)^m + A(y+x)^{m-1} + \ldots = (y+\overline{x-a})(y+\overline{x-b})\ldots(y+\overline{x-n}),$$

in which the terms $\overline{x-a}$, $\overline{x-b}$, may be regarded as single, and hence the factors of the second member as binomials. Now the terms of the first member of this equation, developed and arranged with reference to the ascending powers of y, will give

$$X + X_1 y + \frac{X_2}{2} y^2 + \ldots + \frac{X_{m-1}}{1\cdot 2\ldots(m-1)} y^{m-1} + y^m.$$

And if the second member be developed, and arranged in the same manner, then by **424**, 5, the coefficient of y^0 will be

$$(x-a)(x-b)\ldots(x-m)(x-n).$$

The coefficient of y must be the algebraic sum of the products of the factors $x - a$, $x - b$, etc., taken $m - 1$ in a set.

The coefficient of y^2 must be the algebraic sum of the products of these factors taken $m - 2$ in a set.

In short, these coefficients may all be formed according to the law which governs the product of any number of binomial factors.

But the coefficients of the like powers of y in these two developments must be equal (**368**, III). Hence,

$$X = (x-a)(x-b)(x-c)\ldots\ldots(x-m)(x-n);$$

and since the sum of all the products that can be formed by multiplying m factors in sets of $m-1$ and $m-1$, is the same as the sum of all the quotients which can be obtained by dividing the continued product of the factors by each factor separately, it follows that

$$X_1 = \frac{X}{x-a} + \frac{X}{x-b} + \ldots\ldots + \frac{X}{x-m} + \frac{X}{x-n}.$$

So likewise the sum of the products of the binomial factors taken $m-2$ and $m-2$, is the same as the sum of all the quotients obtained by dividing the continued product by all the different products of the binomial factors taken 2 and 2; that is,

$$\frac{X_2}{2} = \frac{X}{(x-a)(x-b)} + \frac{X}{(x-a)(x-c)} + \ldots\ldots + \frac{X}{(x-a)(x-n)}.$$

By like reasoning it may be shown that

$$\frac{X_3}{2\cdot 3} = \frac{X}{(x-a)(x-b)(x-c)} + \ldots\ldots + \frac{X}{(x-a)(x-m)(x-n)};$$

and so for the next coefficient in order, etc., etc.

EQUAL ROOTS.

435. It has been seen (**427**) that if $a, b, c, \ldots\ldots, m, n$ are the roots of the equation,

$$X = x^m + Ax^{m-1} + Bx^{m-2} + \ldots\ldots + Tx + U = 0,$$

it may be written,

$$X = (x-a)(x-b)(x-c)\ldots\ldots(x-m)(x-n) = 0.$$

Now if a number p of these roots are each equal to a, a number q equal to b, and a number r equal to c, the last equation becomes

$$X = (x-a)^p(x-b)^q(x-c)^r\ldots\ldots(x-m)(x-n) = 0.$$

But since X contains p factors equal to $x-a$, q factors equal to $x-b$, r factors equal to $x-c$, its first derived polynomial will contain the term $\dfrac{X}{x-a}$ p times, the term $\dfrac{X}{x-b}$ q times, the

term $\dfrac{X}{x-c}$ r times, besides the terms $\dfrac{X}{x-m}$, $\dfrac{X}{x-n}$, etc., corresponding to the single roots (**434**); that is,

$$X_1 = \frac{pX}{x-a} + \frac{qX}{x-b} + \frac{rX}{x-c} + \ldots + \frac{X}{x-m} + \frac{X}{x-n}.$$

The factor $(x-a)^p$ is found in every term of this expression for X_1 except the first, from which one of the p equal factors, $x-a$, has been suppressed by division. Hence, $(x-a)^{p-1}$ is the highest power of $x-a$, which is a factor common to all the terms of X_1.

For like reasons $(x-b)^{q-1}$, $(x-c)^{r-1}$ are the highest powers of the factors $x-b$, $x-c$, which are common to all the terms of X_1; hence,

$$(x-a)^{p-1}(x-b)^{q-1}(x-c)^{r-1},$$

is the greatest common divisor which exists between the first member of the proposed equation and its first derived polynomial.

The supposition that the given equation contains one or more sets or species of equal roots, necessarily leads to the existence of this greatest common divisor. Conversely:—If there be a common divisor between X and X_1 there must be one or more sets of equal roots belonging to the equation.

For, if $(x-a)^t$ be a factor of the greatest common divisor, then the composition of X_1 shows that $(x-a)^{t+1}$ is a factor of X, and that a is therefore $t+1$ times a root of the equation $X=0$. Hence the conclusions:

1. An equation involving but one unknown quantity, x, and of which the second member is zero, has equal roots if there be between its first member, X, and its first derived polynomial X_1, a common divisor containing x.

2. The greatest common divisor, D, of X and X_1, is the product of those binomial factors of X, of the first degree with respect to x, which correspond to the equal roots, each raised to a power whose exponent is one less than that with which it enters X. Therefore,

To determine whether an equation has equal roots, and if so, to find them, if possible, we have the following

RULE.—I. *Seek the greatest common divisor between the first member of the proposed equation and its first derived polynomial.*

If no common divisor be found, there are no equal roots; but if one be found, there are equal roots; in which case,

II. *Make an equation by placing the greatest common divisor,* D, *equal to zero; then any quantity which is once a root of* $D = 0$ *will be twice a root of* $X = 0$; *any quantity which is twice a root of* $D = 0$ *will be three times the root of* $X = 0$; *and so on.*

It will at once be seen that, if D contains a factor of the form $(x-a)^t$, t being a positive whole number greater than unity, and we denote the greatest common divisor which exists between D and its first derived polynomial D_1, by D', then D' will contain the factor $(x-a)^{t-1}$. And, again, denoting by D'_1 the first derived polynomial of D', and by D'' their greatest common divisor, $(x-a)^{t-2}$ will be a factor of D''. This process being continued, as the exponent of $(x-a)$,—and, consequently, the degree of the greatest common divisor,—diminishes by one for each operation, it is plain that when the degree of the equation,

$$D = 0,$$

is too high to be solved, we may in certain cases make the determination of the equal roots depend upon the solution of equations of lower degrees, until finally one is obtained which can be solved.

To illustrate, suppose that for the equation,

$$X = 0,$$

it is found that $\quad D = (x-a)^n (x-b)^n (x-c)$;

then $\quad D' = (x-a)^{n-1} (x-b)^{n-1}$,

$$D'' = (x-a)^{n-2} (x-b)^{n-2},$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

$$D^{(n-1)} = (x-a)(x-b).$$

The equation,

$$D^{(n-1)} = (x-a)(x-b) = 0,$$

may be solved, giving the roots $x = a$, $x = b$, and

$$(x-a)^{n+1}, \quad (x-b)^{n+1}, \quad (x-c)^2,$$

are factors of X, or a and b are each $n+1$ times roots, and c twice a root, of the equation,

$$X = 0.$$

Dividing the given equation by the product,

$$(x-a)^{n+1} (x-b)^{n+1} (x-c)^2,$$

its degree will be *depressed* $2n + 4$ units.

EXAMPLES.

1. Does the equation $x^4 - 2x^3 - 7x^2 + 20x - 12 = 0$, contain equal roots, and if so, what are they?

The first derived polynomial of the first member is

$$4x^3 - 6x^2 - 14x + 20.$$

The greatest common divisor between this and the first member of this equation is $x - 2$; therefore 2 is twice a root of the equation, and

$$x^4 - 2x^3 - 7x^2 + 20x - 12$$

may be divided twice by $x - 2$, or once by $(x-2)^2 = x^2 - 4x + 4$. Performing the division, we find the quotient to be $x^2 + 2x - 3$, and the original equation may now be written

$$(x^2 - 4x + 4)(x^2 + 2x - 3) = 0.$$

This equation will be satisfied by the values of x found by placing each of these factors equal to zero. From the first we get $x = 2$, $x = 2$, and from the second $x = 1$, $x = -3$; hence the four roots of the given equation are 1, 2, 2, -3.

2. Find the equal roots of the equation

$$x^5 + 2x^4 - 11x^3 - 8x^2 + 20x + 16 = 0.$$

Ans. 2 and 2; -1 and -1.

3. What are the equal roots of the equation

$$x^5 - 2x^4 + 3x^3 - 7x^2 + 8x - 3 = 0?$$

Ans. It has three roots, each equal to 1.

4. What are the roots of the equation

$$x^4 - 2x^3 - 11x^2 + 12x + 36 = 0?$$

Ans. Its roots are 3, 3, -2, -2.

5. What are the roots of the equation

$$X = x^7 - 5x^6 - 2x^5 + 38x^4 - 31x^3 - 61x^2 + 96x - 36 = 0?$$

We find

$$X_1 = 7x^6 - 30x^5 - 10x^4 + 152x^3 - 93x^2 - 122x + 96,$$

$$D = x^4 - 3x^3 - 3x^2 + 11x - 6.$$

$$D_1 = 4x^3 - 9x^2 - 6x + 11,$$

$$D' = x - 1.$$

Hence 1 is twice a root of the equation $D = 0$, and three times a root of the given equation.

Dividing $D = x^4 - 3x^3 - 3x^2 + 11x - 6$ by $D^2 = x^2 - 2x + 1$, we find for the quotient $x^2 - x - 6 = (x-3)(x+2)$. Therefore,

$$D = (x-3)(x+2)(x-1)^2,$$

and
$$X = (x-3)^2(x+2)^2(x-1)^3.$$

Hence the roots of the given equation are

Ans. $3,\ 3,\ -2,\ -2,\ +1,\ +1,\ +1.$

436. Having an equation involving but one unknown quantity, to transform it into another, the roots of which shall differ from those of the proposed equation by a constant quantity.

Assume $\quad x^m + Ax^{m-1} + Bx^{m-2} + Cx^{m-3} + \ldots\ldots + Tx + U = 0,$

and denote the new unknown quantity by y, and by x' the arbitrary but fixed difference which is to exist between the corresponding values of x and y; we shall then have $x = y + x'$.

Substituting this value of x in the given equation, it becomes

$$(y+x')^m + A(y+x')^{m-1} + B(y+x')^{m-2} + C(y+x')^{m-3} + \ldots\ldots + T(y+x') + U = 0.$$

Developing the terms separately, by the binomial formula, and arranging the aggregate of the results with reference to the ascending powers of y, we have

$$\begin{array}{r|r|r|l}
x'^m & y^0 + \qquad mx'^{m-1} & y + \qquad m\dfrac{m-1}{2}x'^{m-2} & y^2 + \ldots\ldots \\
+ Ax'^{m-1} & + A(m-1)x'^{m-2} & & \\
+ Bx'^{m-2} & + B(m-2)x'^{m-3} & + A(m-1)\dfrac{m-2}{2}x'^{m-3} & \\
+ Cx'^{m-3} & + C(m-3)x'^{m-4} & & \\
\ldots\ldots & \ldots\ldots\ldots & + B(m-2)\dfrac{m-3}{2}x'^{m-4} & \\
+ Tx' & & & \\
+ U & & \ldots\ldots\ldots\ldots &
\end{array}$$

$$\left.\begin{array}{r|l|l}
+ m\dfrac{m-1}{2}x'^2 & y^{m-2} + mx' & y^{m-1} + y^m \\
+ A(m-1)x' & \qquad + A & \\
+ B & &
\end{array}\right\} = 0 \quad \ldots \quad (1).$$

An examination of this developed first member leads to these conclusions:

1. The absolute term of the transformed equation, or the coefficient of y^0, is what the first member of the given equation becomes when x' is substituted for x.

2. The coefficient of y, the first power of the unknown quantity, is what the first derived polynomial of the first member of the given equation becomes, when in it x' takes the place of x.

3. The coefficient of y^2 is what the second derived polynomial of the first member of the given equation becomes when it is divided by 2, and x' takes the place of x.

4. And in general, the coefficient of y^n is what the n^{th} derived polynomial of the first member of the given equation becomes when it is divided by the product of the natural numbers from 1 to n inclusive, and x is replaced by x'.

Representing the first member of the given equation, and its successive derived polynomials, after x' has been substituted for x, by X', X'_1, X'_2, X'_3, etc., respectively, the transformed equation may be written

$$X' + X'_1 y + \frac{X'_2}{1\cdot 2}y^2 + \dots + \frac{X'_{m-2}}{1\cdot 2 \,..\, (m-2)}y^{m-2} + \frac{X'_{m-1}}{1\cdot 2 \,..\, (m-1)}y^{m-1} + y^m = 0.$$

Or, by inverting the order of terms,

$$y^m + \frac{X_{m-1}}{1\cdot 2 \,..\, (m-1)}y^{m-1} + \frac{X'_{m-2}}{1\cdot 2 \,..\, (m-2)}y^{m-2} + \,..\, X'_1 y + X' = 0 \,..\, (1').$$

437. By comparing eqs. (1) and (1′) of the preceding article it is seen that,

$$\frac{X'_{m-1}}{1\cdot 2 \,..\, (m-1)} = mx' + A \quad \dots\dots\dots \quad (2),$$

and $$\frac{X'_{m-2}}{1\cdot 2 \,..\, (m-2)} = m\frac{m-1}{2}x'^2 + A(m-1)x' + B \quad . \quad (3);$$

the degree of the coefficients of equation (1), with respect to x', increasing by at least one from term to term as we pass from left to right, the absolute term being of the m^{th} degree.

Now, since x' is an arbitrary quantity, such a value may be assumed for it as will cause it to satisfy any reasonable condition. We may therefore form an equation, by placing any one of these coefficients equal to zero, regarding x' as the unknown quantity, and any root of this equation will cause the corresponding term of the transformed equation (1′) (**436**), to disappear.

Suppose $\quad mx' + A = 0$; whence $x' = -\dfrac{A}{m}$.

If this value of x' be substituted in the equation just referred to, it takes the form

$$y^m + \frac{X'_{m-2}}{1 \cdot 2 \ldots (m-2)} y^{m-2} + \ldots . X'_1 y + X' = 0.$$

Hence, to transform an equation into another which shall be incomplete in respect to the second term: *Substitute for the unknown quantity another, minus the coefficient of the second term divided by the exponent of the degree of the equation.*

438. The third term will disappear from the transformed equation when x' is made equal to either of the roots of the equation,

$$m\frac{m-1}{2}x'^2 + A(m-1)x' + B = 0.$$

But there may exist such a relation between m, A, and B, that the value, $x' = -\dfrac{A}{m}$, will satisfy this equation; in which case the vanishing of the second term of the transformed equation will involve that of the third. To find what this relation is, substitute this value of x' in the above equation, and it becomes

$$m \cdot \frac{m-1}{2} \cdot \frac{A^2}{m^2} - (m-1)\frac{A^2}{m} + B = 0.$$

This, reduced as follows,

$$\frac{m-1}{2} \cdot \frac{A^2}{m} - (m-1)\frac{A^2}{m} + B = 0,$$

$$(m-1)A^2 - 2(m-1)A^2 + 2mB = 0,$$

$$(m-1)A^2 = 2mB,$$

gives finally $\quad A^2 = \dfrac{2mB}{m-1}$.

When the values m, A, and B, will satisfy this equation, the third term of the transformed equation will disappear with the second. In general, to find the value of x' which will free the transformed equation of the third term, an equation of the second degree must be solved; and to free it of the fourth term, the equation to be solved would be of the third degree; and finally, to make the absolute term disappear, would require the solution of the original equation.

EXAMPLES.

1. Transform the equation $x^2 + 2px - q = 0$, into another which shall not contain the second term.

This is done by making $x = y - \frac{2p}{2} = y - p$ (**437**); whence, by **436**,

$$\begin{aligned} X' &= (p)^2 - 2p\,(p) - q = -q - p^2, \\ X'_1 &= 2\,(p) - 2p = 0, \\ \frac{X'_2}{2} &= \frac{2}{2} = 1. \end{aligned}$$

Therefore, the required equation is

$$y^2 - q - p^2 = 0,$$

from which we find $y = \pm\sqrt{q + p^2}$; and since $x = y - p$, the values of x are given by the formula,

$$x = -p \pm \sqrt{q + p^2},$$

the same as that found by the rule for quadratics.

2. Transform the equation $x^3 + px^2 + qx + r = 0$, into one not having the second term.

Make $x = y - \frac{p}{3}$; then

$$X' = \left(-\frac{p}{3}\right)^3 + p\left(-\frac{p}{3}\right)^2 + q\left(-\frac{p}{3}\right) + r = \frac{2p^3}{27} - \frac{pq}{3} + r,$$

$$X'_1 = 3\left(-\frac{p}{3}\right)^2 + 2p\left(-\frac{p}{3}\right) + q = -\frac{p^2}{3} + q,$$

$$\frac{X'_2}{2} = \frac{2\cdot 3\left(-\frac{p}{3}\right)}{2} + \frac{2p}{2} = 0,$$

$$\frac{X'_3}{2\cdot 3} = \frac{2\cdot 3}{2\cdot 3} = 1.$$

Hence, the equation sought is

$$y^3 - \left(\frac{p^2}{3} - q\right)y + \frac{2p^3}{27} - \frac{pq}{3} + r = 0;$$

or, by making $\frac{p^2}{3} - q = m$, and $\frac{2p^3}{27} - \frac{pq}{3} + r = n$,

$$y^3 - my + n = 0.$$

3. Transform the equation

$$x^4 - 12x^3 + 17x^2 - 9x + 7 = 0,$$

into another which shall not contain the 3d power of the unknown quantity.

By **437**, put $x = y + \frac{12}{4}$; or $x = 3 + y$.

Here $x' = 3$ and $m = 4$.

$X' = (3)^4 - 12\,(3)^3 + 17\,(3)^2 - 9\,(3) + 7$, or $X' = -110$,

$X'_1 = 4\,(3)^3 - 36\,(3)^2 + 34\,(3) - 9$, or $X'_1 = -123$,

$\frac{X'_2}{2} = 6\,(3)^2 - 36\,(3) + 17$, or $\frac{X'_2}{2} = -37$,

$\frac{X'_3}{2 \cdot 3} = 4\,(3)^1 - 12$, or $\frac{X'_3}{2 \cdot 3} = 0$.

Therefore, the transformed equations must be

$$y^4 - 37y^2 - 123y - 110 = 0.$$

4. Transform the equation

$$x^3 - 6x^2 + 13x - 12 = 0,$$

into another wanting its second term.

Put $x = 2 + y$; then

$X' = (2)^3 - 6\,(2)^2 + 13\,(2) - 12$, or $X' = -2$,

$X'_1 = 3\,(2)^2 - 12\,(2) + 13$, or $X'_1 = +1$,

$\frac{X'_2}{2} = 3\,(2)^1 - 6$, or $\frac{X'_2}{2} = 0$,

$\frac{X'_3}{2 \cdot 3} = 1$, or $\frac{X'_3}{2 \cdot 3} = 1$.

Therefore, the transformed equation must be

$$y^3 + y - 2 = 0.$$

5. Transform the equation

$$x^4 - 4x^3 - 8x + 32 = 0,$$

into another whose roots shall be less by 2.

But $x = 2 + y$. *Ans.* $y^4 + 4y^3 - 24y = 0$.

As this transformed equation has no term independent of y, 0 is one of its roots; 2 is therefore a root of the original equation.

6. Transform the equation,

$$x^4 + 16x^3 + 99x^2 + 228x + 144 = 0,$$

into another whose roots shall be greater by 3.

Ans. $y^4 + 4y^3 + 9y^2 - 42y = 0.$

7. Transform the equation,

$$x^4 - 8x^3 + x^2 + 82x - 60 = 0,$$

into one incomplete in respect to its second term.

Ans. $y^4 - 23y^2 + 22y + 60 = 0.$

439. Resuming the transformed equation (1′) (**436**), which is

$$y^m + \frac{X'_{m-1}}{1 \cdot 2 \ldots (m-1)} y^{m-1} + \ldots + \frac{X'_3}{2 \cdot 3} y^3 + \frac{X'_2}{2} y^2 + X'_1 y + X' = 0,$$

and replacing y by its value, $y = x - x'$, it becomes

$$\left.\begin{aligned} &(x - x')^m + \frac{X'_{m-1}}{1 \cdot 2 \ldots (m-1)} (x - x')^{m-1} + \ldots . \\ &+ \frac{X'_3}{2 \cdot 3} (x - x')^3 + \frac{X'_2}{2} (x - x')^2 + X'_1 (x - x') + X' \end{aligned}\right\} = 0.$$

Now it is evident that, by developing the first member of this equation and arranging the result with reference to the descending powers of x, the first member of the original equation will be reproduced; for, by this operation we will have merely retraced the steps by which eq. (1′) was derived from eq. (1) in the article referred to. Hence we have the identical equation,

$$\left.\begin{aligned} &x^m + Ax^{m-1} + Bx^{m-2} + \ldots . \; Sx^2 + Tx + U \\ &= (x - x')^m + \frac{X'_{m-1}}{1 \cdot 2 \ldots (m-1)} (x - x')^{m-1} + \ldots . \\ &+ \frac{X'_3}{2 \cdot 3} (x - x')^3 + \frac{X'_2}{2} (x - x')^2 + X'_1 (x - x') + X' \end{aligned}\right\} \ldots (1).$$

The quotients and remainders obtained by the division of the first member of this equation by any quantity, will not differ from those arising from the division of the second member by the same quantity. Dividing the second member by $x - x'$, the first remainder is X', and the quotient,

$$(x - x')^{m-1} + \frac{X'_{m-1}}{1 \cdot 2 \ldots (m-1)} (x - x')^{m-2} + \ldots .$$
$$+ \frac{X'_3}{2 \cdot 3} (x - x')^2 + \frac{X'_2}{2} (x - x') + X'_1;$$

and this divided again by $x - x'$, will give for the second remainder X'_1 and the quotient,

$$(x-x')^{m-2}+\frac{X'_{m-1}}{1\cdot 2\,..\,(m-1)}(x-x')^{m-3}+\,..\,+\frac{X'_3}{2\cdot 3}(x-x')+\frac{X'_2}{2}.$$

It is unnecessary to continue this process further, to see that these successive remainders are the coefficients of the transformed equation (1') beginning with the absolute term, or the coefficient of y^0. The divisor to be employed is $x - x'$ if the roots of the transformed equation are to be less, in value, than those of the given equation by the constant difference x'; if greater, the divisor must be $x + x'$. Hence, an equation may be transformed into another of which the roots are greater, or less, than those of the given equation by the following

RULE.—I. *Divide the first member of the given equation (the second member being zero) by* x *plus the constant difference between the roots of the two equations, continuing the operation until a remainder is obtained which is independent of* x; *then divide the quotient of this division by the same divisor, and so on, until* m *divisions have been performed.*

II. *Write the transformed equation, making these successive remainders the coefficients of the different powers of the unknown quantity, beginning with the zero power.*

It must be borne in mind that the term *plus* in this rule is used in its algebraic sense.

By a little reflection, it will be seen that the m^{th} quotient will be the coefficient of x^m in the original equation, and that this will also be the coefficient of the highest power of the unknown quantity in the transformed equation.

EXAMPLES.

1. Transform the equation,

$$x^4 - 4x^3 - 8x + 32 = 0,$$

into another of which the roots shall be less by 2.

This is example 5 of the last article. Make

$$x = 2 + y, \text{ or } y = x - 2;$$

then the operation is as follows:

$$\begin{array}{l} x-2\,)\,x^4-4x^3-8x+32\,(\,x^3-2x^2-4x-16 \\ \quad\; \underline{x^4-2x^3} \\ \qquad -2x^3-8x \\ \qquad \underline{-2x^3+4x^2} \\ \qquad\qquad -4x^2-8x \\ \qquad\qquad \underline{-4x^2+8x} \\ \qquad\qquad\qquad -16x+32 \\ \qquad\qquad\qquad \underline{-16x+32} \\ \qquad\qquad\qquad\qquad 0 = X' \end{array}$$

$$\begin{array}{l} x-2\,)\,x^3-2x^2-4x-16\,(\,x^2-4 \\ \quad\; \underline{x^3-2x^2} \\ \qquad\qquad -4x-16 \\ \qquad\qquad \underline{-4x+\;\;8} \\ \qquad\qquad\quad -24 = X'_1 \end{array}$$

$$\begin{array}{l} x-2\,)\,x^2-4\,(\,x+2 \\ \quad\; \underline{x^2-2x} \\ \qquad -2x-4 \\ \qquad \underline{-2x-4} \\ \qquad\qquad 0 = \dfrac{X'_2}{1\cdot 2} \end{array}$$

$$\begin{array}{l} x-2\,)\,x+2\,(\,1 \\ \quad\; \underline{x-2} \\ \qquad\quad 4 = \dfrac{X'_3}{2\cdot 3} \end{array}$$

Hence, the transformed equation is

$$y^4 + 4y^3 \pm 0y^2 - 24y + 0 = 0;$$

or, $$y^4 + 4y^3 - 24y = 0, \text{ as before.}$$

2. Transform the equation, $x^4 - 12x^3 + 17x^2 - 9x + 7 = 0$, into one having roots less by 3.

Here $x = y + 3$, or $y = x - 3$.

OPERATION.

$$\begin{array}{l} x-3\,)\,x^4-12x^3+17x^2-9x+7\,(x^3-9x^2-10x-39 \\ \quad\; \underline{x^4-\;\;3x^3} \\ \qquad -\;\;9x^3+17x^2 \\ \qquad \underline{-\;\;9x^3+27x^2} \\ \qquad\qquad -10x^2-\;\;9x \\ \qquad\qquad \underline{-10x^2+30x} \\ \qquad\qquad\qquad -39x+\;\;\;7 \\ \qquad\qquad\qquad \underline{-39x+117} \\ \qquad\qquad\qquad\qquad -110 = X', \text{ 1st remainder.} \end{array}$$

$$
\begin{array}{l}
x - 3\,)\,x^3 - 9x^2 - 10x - 39\,(\,x^2 - 6x - 28 \\
\quad\quad\;\; \underline{x^3 - 3x^2} \\
\quad\quad\quad\; - 6x^2 - 10x \\
\quad\quad\quad\; \underline{- 6x^2 + 18x} \\
\quad\quad\quad\quad\quad\;\; - 28x - 39 \\
\quad\quad\quad\quad\quad\;\; \underline{- 28x + 84} \\
\quad\quad\quad\quad\quad\quad\quad\; - 123 = X'_1, \text{ 2d remainder.}
\end{array}
$$

$$
\begin{array}{l}
x - 3\,)\,x^2 - 6x - 28\,(\,x - 3 \\
\quad\quad\; \underline{x^2 - 3x} \\
\quad\quad\quad - 3x - 28 \\
\quad\quad\quad \underline{- 3x + \;\, 9} \\
\quad\quad\quad\quad\quad - 37 = \dfrac{X'_2}{2}, \text{ 3d remainder.}
\end{array}
\qquad
\begin{array}{l}
x - 3\,)\,x - 3\,(\,1 \\
\quad\quad\;\;\; \underline{x - 3} \\
\quad\quad\quad\;\; 0 = \dfrac{X'_2 m}{2\cdot 3}, \text{ 4th rem.}
\end{array}
$$

Hence, $y^4 \pm 0y^3 - 37y^2 - 123y - 110 = 0$, is the transformed equation.

We shall have 4 remainders, if we operate on an equation of the 4th degree; 5 remainders with an equation of the 5th degree; and, in general, n remainders with an equation of the n^{th} degree.

The transformation of equations by division, treated of in this article, if performed by the ordinary rule, would be too laborious for practical application; but by a modified method of division, called *Synthetic Division*, it becomes expeditious and easy.

As preliminary to the explanation of this method of division, we must explain the process of

MULTIPLICATION AND DIVISION BY DETACHED COEFFICIENTS.

440. It has been seen that when two polynomials are homogeneous their product is also homogeneous, and the number which denotes its degree is the sum of the numbers denoting the degrees of the factors. It is evident that if the polynomials contain but two letters, and both are arranged with reference to the same letter, the product will be arranged with reference to that letter. Since, in the operation of multiplying the terms of the multiplicand by the terms of the multiplier, the products of the

coefficients are not affected by the literal parts to which they are prefixed, these coefficients may be *detached* and written down with their signs in their proper order, and the multiplication performed as with polynomials. The partial products, numerical or literal, being carefully arranged as if undetached, are then reduced and the literal parts annexed.

EXAMPLES.

1. Multiply $a^2 + 2ax + x^2$ by $a + x$.

OPERATION.

$$\begin{array}{lll}
1 + 2 + 1, & & \text{Detached coefficients of multiplicand.} \\
\underline{1 + 1} & & \text{"} \qquad \text{"} \qquad \text{multiplier.} \\
1 + 2 + 1 & & \\
\underline{\phantom{1+{}}1 + 2 + 1} & & \\
1 + 3 + 3 + 1 & & \text{Coefficients of product.}
\end{array}$$

Now by annexing the proper literal parts to the several terms thus obtained, we have

$$x^3 + 3a^2x + 3ax^2 + x^3, \ Ans.$$

This method of multiplication may be employed when the two polynomials contain but one letter.

2. Multiply $3x^2 - 2x - 1$ by $3x + 2$.

OPERATION.

$$\begin{array}{l}
3 - 2 - 1 \\
\underline{3 + 2} \\
9 - 6 - 3 \\
\underline{+ 6 - 4 - 2} \\
9 \pm 0 - 7 - 2
\end{array}$$

whence

$$9x^3 \pm 0x^2 - 7x - 2,$$

or,

$$9x^3 - 7x - 2, \ Ans.$$

When any of the powers of the letters, between the highest and lowest, do not appear in either factor, the terms corresponding to such powers must be supplied with the coefficient 0.

3. Multiply $x^3 + 2x^2 - 1$ by $x^2 + 2$.

The factors completed are $x^3 + 2x^2 + 0x - 1$ and $x^2 + 0x + 2$. Hence the operation is

$$\begin{array}{r} 1 + 2 + 0 - 1 \\ \underline{1 + 0 + 2 } \\ 1 + 2 + 0 - 1 \\ \underline{2 + 4 + 0 - 2} \\ 1 + 2 + 2 + 3 + 0 - 2 \end{array}$$

and the product,

$$x^5 + 2x^4 + 2x^3 + 3x^2 + 0x - 2,$$

or,

$$x^5 + 2x^4 + 2x^3 + 3x^2 - 2.$$

4. Multiply $3x^2-2x-1$ by $4x+2$. *Ans.* $12x^3-2x^2-8x-2$.

5. Multiply $3x^2-5x-10$ by $2x-4$. *Ans.* $6x^3-22x^2+40$.

6. Multiply x^2+xy+y^2 by x^2-xy+y^2. *Ans.* $x^4+x^2y^2+y^4$.

7. Multiply $x^3 - 4x^2 + 5x - 2$ by $x^2 + 4x - 3$.
Ans. $x^5 - 14x^3 + 30x^2 - 23x + 6$.

441. Now, if detached coefficients can be used in multiplication, so in like cases, they may be employed for division. When the dividend and divisor contain but two letters and are homogeneous, the degree of the quotient will be the excess of the degree of the dividend over that of the divisor.

EXAMPLES.

1. Divide $a^4-3a^3x-8a^2x^2+18ax^3+16x^4$ by $a^2-2ax-2x^2$.

OPERATION.

$$\begin{array}{r|l} 1 - 3 - 8 + 18 + 16 & 1 - 2 - 2 \\ \cline{2-2} \underline{1 - 2 - 2 } & 1 - 1 - 8 \\ -1 - 6 + 18 + 16 & \\ \underline{-1 + 2 + 2 } & \\ -8 + 16 + 16 & \\ \underline{-8 + 16 + 16} & \end{array}$$

Hence the quotient is

$$a^2 - ax - 8x^2.$$

2. Divide $a^5 - 5a^3b^2 + a^2b^3 + 6ab^4 - 2b^5$ by $a^3 - 3ab^2 + b^3$.

In this example we must supply the term $0 \cdot a^4b$ in the dividend, and the term $0 \cdot a^2b$ in the divisor. The operation then is,

$$\begin{array}{r|l}
1 + 0 - 5 + 1 + 6 - 2 & 1 + 0 - 3 + 1 \\ \cline{2-2}
\underline{1 + 0 - 3 + 1 \qquad\qquad} & 1 + 0 - 2 \\
0 - 2 + 0 + 6 - 2 & \\
\underline{-2 + 0 + 6 - 2} &
\end{array}$$

Therefore we have, for the quotient,

$$a^2 + 0 \cdot ab - 2b^2,$$

or,

$$a^2 - 2b^2.$$

3. Divide $x^5 - 4x^4 - 17x^3 - 13x^2 - 11x - 10$ by $x^2 + 3x + 2$.

OPERATION.

$$\begin{array}{r|l}
1 - 4 - 17 - 13 - 11 - 10 & 1 + 3 + 2 \\ \cline{2-2}
\underline{1 + 3 + 2 \qquad\qquad\qquad\quad} & 1 - 7 + 2 - 5 \\
-7 - 19 - 13 - 11 - 10 & \\
\underline{-7 - 21 - 14 \qquad\quad\;\;} & \\
+\ 2 + \ 1 - 11 - 10 & \\
\underline{+\ 2 + \ 6 + \ 4 \qquad} & \\
-\ 5 - 15 - 10 & \\
\underline{-\ 5 - 15 - 10} &
\end{array}$$

Hence the quotient is

$$x^3 - 7x^2 + 2x - 5.$$

When the dividend and divisor contain but a single letter, absent terms in either, answering to powers of this letter between the highest and lowest, must be inserted with the coefficient 0.

In the examples we have wrought to illustrate the method of division by detached coefficients, the coefficients have been taken entire, that of the first term of the divisor, in each case, being unity; the process, however, will be the same whatever these coefficients may be. When the coefficient of the first term of the divisor is not unity, it may be made so by dividing both dividend and divisor by this coefficient. The quotient term will then be the first term of the corresponding dividend, as is seen in all the above examples.

SYNTHETIC DIVISION.

442. To explain what synthetic division is, and to deduce a rule for executing it, let us take the first example in the preceding article. If the signs of the second and third terms of the divisor be changed, each remainder will be found, by *adding* the terms of the product of these two terms by the term of the quotient, to the corresponding terms of the dividend; observing that by the nature of the operation, the product of the first term of the divisor by the term of the quotient, equals the first term of the dividend. Besides, since the first term of the divisor is unity, any quotient term is the same as the first term of the partial dividend to which it belongs.

The process may now be indicated as follows:

$$\begin{array}{rrrrrr|l} & 1 & -3 & -8 & +18 & +16 & 1+2+2 \\ & & 2 & -2 & -16 & & \\ & & & 2 & -2 & -16 & \\ \hline \text{Quotient,} & 1 & -1 & -8 & 0 & 0 & \end{array}$$

Hence the quotient is $a^2 - ax - 8x^2$, as before found.

The dividend and divisor are written in the usual way, after changing the signs of the last two terms of the latter; and a horizontal line is drawn far enough beneath the dividend for two intervening rows of figures. Bring down the first term of the dividend for the first term of the quotient. The products of the second and third terms of the divisor by the first term of the quotient are written, the first in the first row under the second term of the dividend, and the second in the second row under the third term of the dividend. The sum of the second vertical column is then written for the second term of the quotient. The next step is to multiply the second and third terms of the divisor by the second term of the quotient, placing the first product in the first row under the third term of the dividend, and the second in the second row under the fourth term of the dividend. The sum of the third vertical column is the third term of the quotient. The sums of the fourth and fifth columns each reduce to zero.

The operation for the last example in the preceding article is

$$\begin{array}{rrrrrr|l}
1 & -4 & -17 & -13 & -11 & -10 & \underline{1-3-2} \\
 & -3 & +21 & -6 & +15 & & \\
 & & -2 & +14 & -4 & +10 & \\
\hline
1 & -7 & +2 & -5 & 0 & 0 &
\end{array}$$

and for the quotient we have

$$x^3 - 7x^2 + 2x - 5.$$

No difficulty will now be experienced in understanding this general

RULE.—I. *If the coefficient of the first term of the arranged divisor is not unity, make it so by dividing both dividend and divisor by this coefficient.*

II. *Write down the detached coefficients of the dividend and divisor in the usual way, changing the signs of all the terms of the latter except the first, and draw a line far enough below the dividend for as many intervening rows of figures as there are terms, less one, in the divisor, and bring down the first term of the dividend, regarded as forming a vertical column, for the first term of the quotient.*

III. *Write the products of the second, third, etc., terms of the divisor by the first term of the quotient, beneath the second, third, etc., terms of the dividend in their order, and in the first, second, etc., rows of figures; and bring down the sum of the second vertical column for the second term of the quotient.*

IV. *Multiply the terms of the divisor, exclusive of the first, as before, by the second term of the quotient, and write the products in their respective rows, beneath the terms of the dividend beginning at the third; bring down the sum of the third vertical column for the third term of the quotient.*

V. *Continue this process until a vertical column is found of which the sum is zero, the sums of all the following also being zero when the division is exact; otherwise continue the operation until the desired degree of approximation is attained. Having thus found the coefficients of the quotient, annex to them the proper literal parts.*

In applying this method of division it is unnecessary to write the first term of the divisor, since it is unity and is not used in the operation.

EXAMPLES.

1. Divide $1 - x$ by $1 + x$. *Ans.* $1-2x+2x^2-2x^3+$etc.

2. Divide 1 by $1 + x$. *Ans.* $1-x+x^2-x^3+x^4-$etc.

3. Divide $a^5-5a^4x+10a^3x^2-10a^2x^3+5ax^4-x^5$ by $a^2-2ax+x^2$.
Ans. $a^3 - 3a^2x + 3ax^2 - x^3$.

4. Divide $x^6 - 5x^5 + 15x^4 - 24x^3 + 27x^2 - 13x + 5$ by $x^4 - 2x^3 + 4x^2 - 2x + 1$. *Ans.* $x^2 - 3x + 5$.

5. Divide $x^7 - y^7$ by $x - y$.
Ans. $x^6 + x^5y + x^4y^2 + x^3y^3 + x^2y^4 + xy^5 + y^6$.

443. The transformation of an equation into another having roots less or greater than those of the given equation by a fixed quantity, may now be expeditiously made by the method of synthetic division.

1. Transform the equation $x^4-4x^3-8x+32=0$, into another whose roots shall be less by 2.

The second power of x not appearing in this equation, it must be introduced with ± 0 for its coefficient.

FIRST OPERATION.

$$\begin{array}{l|l}
1 - 4 \pm 0 - 8 + 32 & 2 \\
\quad\; 2 - 4 - 8 - 32 & \\
\hline
1 - 2 - 4 - 16, \quad 0 = X'. &
\end{array}$$

SECOND OPERATION.

$$\begin{array}{l|l}
1 - 2 - 4 - 16 & 2 \\
\quad\; 2 \pm 0 - \;\, 8 & \\
\hline
1 \pm 0 - 4, -24 = X'_1. &
\end{array}$$

THIRD OPERATION.

$$\begin{array}{l|l}
1 \pm 0 - 4 & 2 \\
\quad\; 2 + 4 & \\
\hline
1 + 2, \quad 0 = \dfrac{X'_2}{2}. &
\end{array}$$

FOURTH OPERATION.

$$\begin{array}{l|l}
1 + 2 & 2 \\
\quad\; 2 & \\
\hline
1, + 4 = \dfrac{X'_3}{2 \cdot 3}. &
\end{array}$$

Hence the transformed equation is

$$y^4 + 4y^3 - 24y = 0.$$

Instead of keeping the above operations separated, they may be united and arranged as follows:

$$
\begin{array}{l}
1 - 4 \pm 0 - 8 + 32 \,\underline{|2} \\
\quad\;\; 2 - 4 - 8 - 32 \\
\hline
\quad - 2 - 4 - 16, \quad 0 = X' \\
\quad\;\; 2 \quad 0 - 8 \\
\hline
\quad\;\; 0 - 4, - 24 = X'_1 \\
\quad\;\; 2 + 4 \\
\hline
\quad\;\; 2, \quad 0 = \dfrac{X'_2}{2} \\
\quad\;\; \underline{2} \\
\quad\;\; 4 = \dfrac{X'_3}{2\cdot 3}.
\end{array}
$$

To understand this, it is only to be borne in mind that the divisor is the same throughout, and that the first term, 1, of the successive dividends, which if written would fall in the vertical column at the left, is omitted.

Transform the equation $x^4 - 12x^3 + 17x^2 - 9x + 7 = 0$, into another whose root shall be 3 less.

OPERATION.

$$
\begin{array}{l}
1 - 12 + 17 - 9 + 7 \,\underline{|3} \\
\quad + 3 - 27 - 30 - 117 \\
\hline
\quad - 9 - 10 - 39, - 110 = X' \\
\quad + 3 - 18 - 84 \\
\hline
\quad - 6 - 28, - 123 = X'_1 \\
\quad + 3 - 9 \\
\hline
\quad - 3, - 37 = \dfrac{X'_2}{2} \\
\quad \underline{+ 3} \\
\quad\quad 0 = \dfrac{X'_3}{2\cdot 3}.
\end{array}
$$

Hence the transformed equation is

$$y^4 + 0y^3 - 37y^2 - 123y - 110 = 0.$$

Transform the equation $x^3 - 12x - 28 = 0$, into another whose roots shall be 4 less. Make $x = y + 4$.

OPERATION.

$$
\begin{array}{cccccl}
1 & 0 & -12 & -28 & \underline{|4} & \\
 & \underline{4} & \underline{+16} & \underline{+16} & & \\
 & 4 & 4 & -12 = X' & & \\
 & \underline{4} & \underline{32} & & & \\
 & 8 & 36 = X'_1 & & & \\
 & \underline{4} & & & & \\
 & 12 = \frac{X'_2}{2}. & & & &
\end{array}
$$

Hence the transformed equation must be

$$y^3 + 12y^2 + 36y - 12 = 0.$$

Transform the equation $x^3 - 10x^2 + 3x - 6946 = 0$, into another whose roots shall be less by 20. We make $x = 20 + y$.

OPERATION.

$$
\begin{array}{ccccc}
1 & -10 & 3 & -6946 & \underline{|20} \\
 & \underline{20} & \underline{200} & \underline{4060} & \\
 & 10 & 203 & \underline{\underline{-2886}} & \\
 & \underline{20} & \underline{600} & & \\
 & 30 & \underline{\underline{803}} & & \\
 & \underline{20} & & & \\
 & \underline{\underline{50}} & & &
\end{array}
$$

The three remainders are the numbers just above the *double lines,* which give the following transformed equation:

$$y^3 + 50y^2 + 803y - 2886 = 0.$$

Transform this equation into another whose roots shall be less by 3. Put $y = 3 + z$.

$$
\begin{array}{ccccc}
1 & 50 & 803 & -2886 & \underline{|3} \\
 & \underline{3} & \underline{159} & \underline{+2886} & \\
 & 53 & 962 & \underline{\underline{0}} & \\
 & \underline{3} & \underline{168} & & \\
 & 56 & \underline{\underline{1130}} & & \\
 & \underline{3} & & & \\
 & \underline{\underline{59}} & & &
\end{array}
$$

Hence the transformed equation is

$$z^3 + 59z^2 + 1130z = 0.$$

This equation may be verified by making $z = 0$; which gives

$$y = 3, \quad \text{and} \quad x = 20 + 3 = 23.$$

444. If the signs of the alternate terms of any complete equation involving but one unknown quantity be changed, the signs of all the roots will be changed.

In the general equation

$$x^m + Ax^{m-1} + Bx^{m-2} + \ldots + Tx + U = 0 \quad . \quad . \quad . \quad (1),$$

let the signs follow each other in any order whatever. Changing the signs of the alternate terms of this equation, beginning with the second, we have

$$x^m - Ax^{m-1} + Bx^{m-2} - \ldots \pm Tx \mp U = 0 \quad . \quad . \quad . \quad (2);$$

but if the change begin with the first term, we have

$$-x^m + Ax^{m-1} - Bx^{m-2} + \ldots \mp Tx \pm U = 0 \quad . \quad . \quad . \quad (3).$$

Now, if a be a root of (1), its first member reduces to zero when a is substituted for x; that is, the sum of the positive terms becomes equal to the sum of the negative terms. But if $-a$ be substituted for x in (2) and (3), the numerical values of the terms of these equations will be equal to the values of the corresponding terms of (1), while the signs of the terms in (2), if m is an even number, will be the same, and those of (3), opposite to the signs of the terms of like degree in (1). If m is an odd number, the reverse will be true in respect to signs. In either case, however, if a is a root of (1), $-a$ is a root of both (2) and (3).

An obvious consequence of this proposition is, that the roots of an equation are not affected by changing the signs of all its terms.

EXAMPLES.

1. The roots of the equation $x^3 - 7x^2 + 13x - 3 = 0$, are 3, $2 + \sqrt{3}$, and $2 - \sqrt{3}$; what will be the roots of the equation $x^3 + 7x^2 + 13x + 3 = 0$? *Ans.* -3, $-2 - \sqrt{3}$, $-2 + \sqrt{3}$.

2. The roots of the equation $x^4 - 3x^3 + 3x^2 + 17x - 18 = 0$,

are 1, -2, $2+\sqrt{-5}$, and $2-\sqrt{-5}$; what are the roots of the equation $x^4+3x^3+3x^2-17x-18=0$?

Ans. -1, $+2$, $-2-\sqrt{-5}$, $-2+\sqrt{-5}$.

445. If all the coefficients of an equation are rational, surd roots can enter the equation only by pairs, and if all the coefficients are real, imaginary roots can enter the equation only by pairs.

Let the coefficients A, B, ... U, of the equation,

$$x^m+Ax^{m-1}+Bx^{m-2}+\ldots\ldots Tx+U=0 \quad . \quad . \quad . \quad (1),$$

be all real and rational, and suppose that $a\pm\sqrt{\pm b}$ is one of the roots of this equation.

Substituting this value for x, we have

$$\left.\begin{array}{l}(a\pm\sqrt{\pm b})^m+A(a\pm\sqrt{\pm b})^{m-1}+B(a\pm\sqrt{\pm b})^{m-2}\\ \quad +\ldots\ldots+T(a\pm\sqrt{\pm b})+U\end{array}\right\}=0 \quad . \quad . \quad . \quad (2).$$

Expanding the several terms of this equation by the binomial formula, we have

$$a^m\pm ma^{m-1}.\sqrt{\pm b}+m\frac{m-1}{2}a^{m-2}.(\sqrt{\pm b})^2\pm\ldots\ldots\pm(\sqrt{\pm b})^m,$$

$$Aa^{m-1}\pm A(m-1)a^{m-2}.\sqrt{\pm b}+A(m-1)\frac{m-2}{2}a^{m-3}.(\sqrt{\pm b})^2\pm\ldots\ldots$$
$$\pm(\sqrt{\pm b})^{m-1},$$

$$Ba^{m-2}\pm B(m-2)a^{m-3}.\sqrt{\pm b}+B(m-2)\frac{m-3}{2}a^{m-3}.(\sqrt{\pm b})^2\pm\ldots\ldots$$
$$\pm(\sqrt{\pm b})^{m-2},$$

. . . .

$Ta\pm T\sqrt{\pm b}$,

U;

observing, in reference to the final terms, $\pm(\sqrt{\pm b})^m$, $\pm(\sqrt{\pm b})^{m-1}$, etc., that the sign $\pm$ is to be used before those only which have odd numbers for their exponents; when the exponent is even, the plus sign is to be understood.

If the root of (1) be $a+\sqrt{b}$, the aggregate of these developments will be composed of two parts, the one rational and the other surd. The rational part will be the algebraic sum of those terms which have the even powers of $\sqrt{b}$ for factors, the zero power being included. Represent this part by M.

The irrational part will be the algebraic sum of the terms having the odd powers of $\sqrt{b}$ for factors. But since $(\sqrt{b})^3 = b\sqrt{b}$, $(\sqrt{b})^5 = b^2\sqrt{b}$, etc., the sum of the terms of this part can be represented by a single term of the form $N\sqrt{b}$, N being the algebraic sum of the coefficients of $\sqrt{b}$. Hence (2), under the supposition that $a + \sqrt{b}$ is a root of (1), becomes

$$M + N\sqrt{b} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3);$$

which can be true only when we have separately $M = 0$, $N = 0$ (**272**, 4).

In reducing (2) to (3), the upper signs in the expansions of the terms of (2) were used. If the lower signs in the equation and the expansions of its terms be used,—which is equivalent to supposing $a - \sqrt{b}$ to be a root of (1),—the reduced equation will be

$$M - N\sqrt{b} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4),$$

in which M and N are evidently the same as in (3). Hence if (1) has a root, $a + \sqrt{b}$, it has also the root $a - \sqrt{b}$.

Now let us suppose that $a + \sqrt{-b}$ is a root of (1); then since the even powers of $\sqrt{-b}$ are real and the odd powers imaginary, the developed first member of (2) will be composed of two parts, the one real and the other imaginary. Represent the real part by M'.

The imaginary part is the algebraic sum of the terms having the odd powers of $\sqrt{-b}$ for factors. But since $(\sqrt{-b})^3 = \sqrt{b^2(-b)} = b\sqrt{-b}$, $(\sqrt{-b})^5 = \sqrt{b^4(-b)} = b^2\sqrt{-b}$, etc., the sum of the terms of this part can be represented by a single term of the form $N'\sqrt{-b}$. Hence, under the supposition that $a + \sqrt{-b}$ is a root of (1), (2) becomes

$$M' + N'\sqrt{-b} = 0 \quad . \quad . \quad . \quad . \quad . \quad (5),$$

which requires that we have separately $M' = 0$, $N' = 0$ (**267**). By using the lower signs of the terms and their expansions in (2),—which supposes $a - \sqrt{-b}$ to be a root of (1),—we find

$$M' - N'\sqrt{-b} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (6);$$

and by a simple inspection of the expanded terms of (2), we see that M' and N' in (5) and (6) are the same.

Whence we conclude that if (1) has a root, $a + \sqrt{-b}$, it has also the root, $a - \sqrt{-b}$.

RULE OF DES CARTES.

446. *The number of real positive roots of the equation* $X = 0$ *cannot exceed the number of variations in the signs of its terms; and, if the equation* $X = 0$ *is complete, the number of real negative roots cannot exceed the number of permanences in the signs of its terms.*

NOTE.—In any series of quantities a pair of consecutive like signs is called a *permanence* of signs, and a pair of consecutive unlike signs is called a *variation* of signs. Thus, in the expression $x^8 - 3x^7 - 4x^6 + 7x^5 + 3x^4 + 2x^3 - x^2 - x + 1$, there are four permanences and four variations.

Represent the real positive roots of the equation,

$$X = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1),$$

by $a, b, c \ldots$, and suppose (1) to be divided by the product of all the factors $x - a, x - b, x + c \ldots$, corresponding to the real positive roots. Represent the resulting equation by

$$X_1 = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (2).$$

This equation has no real positive roots.

We shall now show that if (2) be multiplied by the factor $x - a$ corresponding to a real positive root, the number of variations in the resulting equation will be at least one greater than in (2).

1st. Suppose (2) to be complete, and let the signs of its terms be

$$\begin{array}{lcccccc}
\text{terms be} & + & + & - & - & + . & \\
\text{The signs of the multiplier are} & + & - . & & & & \\
\hline
 & + & + & - & - & + & \\
 & & - & - & + & + & - \\
\hline
\text{The signs of the product are} & + & \pm & - & \mp & + & - .
\end{array}$$

A double sign is placed where the sign of any term in the product is ambiguous.

Now, taking the ambiguous signs as we please, the number of variations in the product is greater than in the multiplicand; and this is still true if we suppose some or all of the terms having ambiguous signs to vanish.

2d. If (2) is incomplete, reduce it to a complete form by inserting the missing terms with zero for the coefficient of each; the resulting equation will contain at least as many variations as (2). Multiplying the completed equation by $x - a$, the num-

ber of variations in the product will be greater than in the multiplicand, by what has just been shown. But the product thus obtained is the same as the product of X_1 and $x - a$; hence, the number of variations in the product of X_1 and $x - a$ is greater than in X.

We have thus shown that when the factor $x - a$ is introduced into (2), the resulting equation contains at least one more variation than (2). In like manner it may be shown that when the factor $x - b$ is introduced into the resulting equation, at least one more variation is introduced; and so on.

Hence the number of real positive roots of the equation $X = 0$ cannot exceed the number of variations in the signs of its terms.

We prove the second part of the theorem as follows:

Suppose (1) to be complete, and let the signs of its alternate terms be changed; then the signs of the roots will be changed (444), the permanences will become variations, and the variations will become permanences. But the number of real positive roots of the resulting equation cannot exceed the number of variations in the signs of its terms; hence the number of real negative roots of the given equation cannot exceed the number of permanences in the signs of its terms.

447. Although the introduction of a positive root will always give an additional variation of signs, it is not true that a variation of signs in the terms of an equation necessarily implies the presence of a *real* positive root. Thus, the equation,

$$x^3 - x^2 - 7x + 15 = 0$$

has 2 variations of signs, and 1 permanence. But its roots are

$$2 + \sqrt{-1},\ 2 - \sqrt{-1},\ \text{and}\ -3,$$

no one being positive and real.

But when the roots are all real, the number of positive roots is equal to the number of variations, and the number of negative roots is equal to the number of permanences.

CARDAN'S RULE FOR CUBIC EQUATIONS.

448. It has been shown (**437**), that any equation can be transformed into another which shall be deficient of its second term. That is, every cubic equation can be reduced to the form of

$$x^3 + 3px = 2q \quad . \quad . \quad . \quad . \quad . \quad (1);$$

and the solution of this equation must involve the general solution of cubics. We make $3p$ the coefficient of x, and $2q$ the absolute term, in order to avoid fractions in the following investigations:

Assume $\qquad x = v + y; \qquad$ then (1) becomes

$$(v + y)^3 + 3p(v + y) = 2q \quad . \quad . \quad (2).$$

Expanding and reducing,

$$v^3 + y^3 + 3(vy + p)(v + y) = 2q \quad . \quad (3).$$

Now as the division of x into two parts is entirely arbitrary, we are permitted to assume that

$$vy + p = 0 \quad . \quad . \quad . \quad . \quad . \quad (4);$$

whence by (3), $\qquad v^3 + y^3 = 2q \quad . \quad . \quad . \quad . \quad . \quad (5).$

If we obtain the value of y from (4), and substitute it in (5), we shall have, after reducing,

$$v^6 - 2qv^3 = p^3 \quad . \quad . \quad . \quad . \quad . \quad (6);$$

whence, $\qquad v^3 = q \pm \sqrt{q^2 + p^3} \quad . \quad . \quad . \quad . \quad (7).$

Substituting this value of v^3 in (5),

$$y^3 = q \mp \sqrt{q^2 + p^3} \quad . \quad . \quad . \quad . \quad (8).$$

But by hypothesis $x = v + y$; hence, taking the sum of the cube roots of (7) and (8),

$$x = \left(q + \sqrt{q^2 + p^3}\right)^{\frac{1}{3}} + \left(q - \sqrt{q^2 + p^3}\right)^{\frac{1}{3}} \quad (A),$$

which is Cardan's formula for cubic equations.

449. When p is negative, in the given equation, and its cube numerically greater than q^2, the expression $\sqrt{q^2 + p^3}$ becomes imaginary; this is called the *Irreducible Case.* We must not conclude, however, that in this case the roots of the equation are imaginary; for, admitting the expression $\sqrt{q^2 + p^3}$ to be imaginary, it can be represented by $a\sqrt{-1}$; whence the value of x in formula (A) becomes

$$x = \left(q + a\sqrt{-1}\right)^{\frac{1}{3}} + \left(q - a\sqrt{-1}\right)^{\frac{1}{3}} \quad . \quad . \quad (1);$$

or, $\quad x = q^{\frac{1}{3}}\left(1 + \frac{a}{q}\sqrt{-1}\right)^{\frac{1}{3}} + q^{\frac{1}{3}}\left(1 - \frac{a}{q}\sqrt{-1}\right)^{\frac{1}{3}} \quad . \quad . \quad (2);$

or, $\quad \dfrac{x}{q^{\frac{1}{3}}} = \left(1 + \frac{a}{q}\sqrt{-1}\right)^{\frac{1}{3}} + \left(1 - \frac{a}{q}\sqrt{-1}\right)^{\frac{1}{3}} \quad . \quad . \quad (3).$

Now by actually expanding the two parts in the second member of (3), and adding the results, the terms containing $\sqrt{-1}$ will disappear and the final result will be real. In the irreducible case *all* the roots of the equation are real; formula (A) is therefore practically applicable only when two of the roots are imaginary. In this case the real root can be found directly by the formula; the equation may then be depressed, by division, to a quadratic, which will give the two imaginary roots.

EXAMPLES.

1. Find the roots of the equation,

$$x^3 - 7x^2 + 14x - 20 = 0.$$

To transform this equation into another deficient of its 2d term, according to (**437**), put $x = y + \frac{7}{3}$; and we shall have for the transformed equation,

$$y^3 - \tfrac{7}{3}y = \tfrac{344}{27}.$$

To apply the formula to this equation, we have

$$3p = -\tfrac{7}{3}, \text{ or } p = -\tfrac{7}{9};$$
$$2q = \tfrac{344}{27}, \text{ or } q = \tfrac{172}{27}.$$
$$\sqrt{q^2 + p^3} = \sqrt{(\tfrac{172}{27})^2 - (\tfrac{7}{9})^3} = \pm \tfrac{171}{27}.$$
$$y = (\tfrac{172}{27} \pm \tfrac{171}{27})^{\frac{1}{3}} + (\tfrac{172}{27} \mp \tfrac{171}{27})^{\frac{1}{3}} = \tfrac{7}{3} + \tfrac{1}{3} = \tfrac{8}{3}.$$
$$x = \tfrac{8}{3} + \tfrac{7}{3} = 5, \text{ the real root.}$$

Dividing the given equation by $x - 5$, we obtain for the depressed equation

$$x^2 - 2x + 4 = 0;$$

whence, $\quad x = 1 \pm \sqrt{-3}.$

Hence the three roots are 5, $1 + \sqrt{-3}$, and $1 - \sqrt{-3}$.

2. Given $x^3 + 6x = 88$, to find the values of x.

To apply the formula, we have

$$3p = 6, \qquad \text{or } p = 2;$$
$$2q = 88, \qquad \text{or } q = 44.$$

whence, $\quad \sqrt{q^2 + p^3} = \sqrt{1936 + 8} = \pm 44.090815+.$

And we have

$$x = (44 + 44.090815)^{\frac{1}{3}} + (44 - 44.090815)^{\frac{1}{3}};$$

or, $\quad x = 4.4495 - .4495 = 4$, the real root.

The depressed equation will be $x^2 + 4x - 22 = 0$; whence,

$$x = -2 \pm 3\sqrt{-2};$$

and the three roots are 4, $-2 + 3\sqrt{-2}$, and $-2 - 3\sqrt{-2}$.

3. Given $x^3 - 6x = 5.6$, to find one value of x.

This example presents the irreducible case; the solution, by the method of series, is as follows:

We have $\quad p = -2,\ q = 2.8$; hence,

$$x = (2.8 + \sqrt{7.84 - 8})^{\frac{1}{3}} + (2.8 - \sqrt{7.84 - 8})^{\frac{1}{3}};$$

or, $\quad x = (2.8 + .4\sqrt{-1})^{\frac{1}{3}} + (2.8 - .4\sqrt{-1})^{\frac{1}{3}};$

or, $\quad \dfrac{x}{\sqrt[3]{2.8}} = (1 + \frac{1}{7}\sqrt{-1})^{\frac{1}{3}} + (1 - \frac{1}{7}\sqrt{-1})^{\frac{1}{3}}.$

Put $\quad b = \frac{1}{7}\sqrt{-1}$; then $b^2 = -\frac{1}{49}$, $b^4 = \frac{1}{49} \times \frac{1}{49}$.

Also, $(1 + \frac{1}{7}\sqrt{-1})^{\frac{1}{3}} = (1 + b)^{\frac{1}{3}}$; $(1 - \frac{1}{7}\sqrt{-1})^{\frac{1}{3}} = (1 - b)^{\frac{1}{3}}$.

By the binomal theorem

$$(1 + b)^{\frac{1}{3}} = 1 + \frac{1}{3}b - \frac{2}{3 \cdot 6}b^2 + \frac{2 \cdot 5}{3 \cdot 6 \cdot 9}b^3 - \frac{2 \cdot 5 \cdot 8}{3 \cdot 6 \cdot 9 \cdot 12}b^4 + \ldots,$$

$$(1 - b)^{\frac{1}{3}} = 1 - \frac{1}{3}b - \frac{2}{3 \cdot 6}b^2 - \frac{2 \cdot 5}{3 \cdot 6 \cdot 9}b^3 - \frac{2 \cdot 5 \cdot 8}{3 \cdot 6 \cdot 9 \cdot 12}b^4 - \ldots.$$

Sum $\quad = 2 - 2\left(\dfrac{2}{3 \cdot 6}b^2\right) - 2\left(\dfrac{2 \cdot 5 \cdot 8}{3 \cdot 6 \cdot 9 \cdot 12}b^4\right) - \ldots.$

$$= 2 + .004535 + .000034 = 2.004501.$$

Hence,

$$\frac{x}{\sqrt[3]{2.8}} = 2.004569;\ x = (2.004569)\sqrt[3]{2.8} = 2.82535, \textit{ Ans.}$$

4. Given $x^3 - 6x - 6 = 0$, to find one value of x.

Ans. $x = \sqrt[3]{2} + \sqrt[3]{4} = 2.8473 +.$

5. Given $x^3 + 9x - 6 = 0$, to find one value of x.

Ans. $x = \sqrt[3]{9} + \sqrt[3]{-3} = .63783 +.$

6. Given $x^3 + 6x^2 - 13x + 24 = 0$, to find the values of x.

Ans. $x = -8, 1 + \sqrt{-2}$, or $1 - \sqrt{-2}$.

SECTION IX.

SOLUTION OF NUMERICAL EQUATIONS OF HIGHER DEGREES.

LIMITS OF REAL ROOTS.

450. All positive roots of an equation are comprised between 0 and $+\infty$, and all negative roots between 0 and $-\infty$. But in the solution of numerical equations of higher degrees, it is necessary to be able at once to assign much narrower limits. As preliminary to this, we will first show how an equation is affected by substituting for the unknown quantity numbers greater or less than the roots, and numbers between which the roots are comprised.

451. If an equation, in its general form, be regarded as the product of the binomial factors formed by annexing the roots, with their opposite signs, to x, we observe that the *sign* of this product cannot be affected by the imaginary roots. For, according to **445**, if an equation have one root in the form of $a+\sqrt{-b}$, it will have another in the form of $a-\sqrt{-b}$. But we have

$$\left(x-a-\sqrt{-b}\right)\left(x-a+\sqrt{-b}\right)=(x-a)^2+b,$$

a result which is in all cases positive.

452. Let a, b, c, d, etc., be the real roots of an equation, arranged in the order of their algebraic values; then the equation may be represented as follows:

$$(x-a)(x-b)(x-c)(x-d)\ldots\ldots=0.$$

If we substitute h for x, the first member will become

$$(h-a)(h-b)(h-c)(h-d)\ldots\ldots$$

Now if h be less than the least root, a, every factor will be negative; and the whole product will be positive or negative, according as the number of factors is even or odd. But the number of factors is equal to the degree of the equation (**427**); hence,

1. *If a number less than the least root be substituted for* x *in an*

equation, the result will be positive when the equation is of an even degree, and negative when the equation is of an odd degree.

Again, if h be greater than the greatest root, then every factor will be positive, and consequently the whole product positive. Hence,

2. *If a number greater than the greatest root be substituted for* x *in an equation, the result will in all cases be positive.*

Still again; suppose h to be at first less than a, but afterward greater than a and less than b. This change in the value of h will change the sign of the factor $(h-a)$, and consequently the sign of the whole product. If in the next place h be made greater than b but less than c, the sign of the product will be changed again, for the same reason as before. And in general, there must be a change of sign every time the variable h passes the value of a real root of the equation, and at no other time. Hence,

3. *If when two numbers are substituted in succession for* x, *in an equation, the results have contrary signs, there must be at least one real root included between these numbers.*

It may be observed, also, that if *one, three, five*, or any *odd* number of roots be included between the two numbers substituted, the results will show a change of signs. But if an *even* number of roots be included, there will be no change of signs.

453. If P denote the numerical value of the greatest negative coefficient in an equation, and n the number of terms which precede the first negative coefficient, then $\sqrt[n]{P}+1$ will be a superior limit of the positive roots of this equation.

Let $x^m + Ax^{m-1} + Bx^{m-2} + Cx^{m-3} + \ldots + Tx + U = 0 \ldots (1).$

If we omit those positive terms, if any, which occur between x^m and the first negative term, and then put $-P$ for the coefficient of every other term after x^m, the first member of (1) becomes

$$x^m - (Px^{m-n} + Px^{m-n-1} + \ldots + Px + P) \ldots . (2).$$

Now it is evident that any value which, substituted for x, will give a positive result in (2) will give a positive result also in (1). For, the sum of all the negative terms in (1) cannot possibly be greater than the negative part of (2); besides, there may be one or more positive terms in (1) which are omitted in (2).

Dividing every term of (2) by x^m, we obtain

$$1-\left(\frac{P}{x^n}+\frac{P}{x^{n+1}}+\frac{P}{x^{n+2}}+\ldots+\frac{P}{x^{m-1}}+\frac{P}{x^m}\right) \quad . \quad . \quad . \quad (3).$$

Make $x=\sqrt[n]{P}+1=r+1$, where r is put in the place of $\sqrt[n]{P}$ for the sake of simplicity. Remembering that $P=r^n$, we have

$$1-\left(\frac{r^n}{(r+1)^n}+\frac{r^n}{(r+1)^{n+1}}+\ldots+\frac{r^n}{(r+1)^{m-1}}+\frac{r^n}{(r+1)^m}\right).$$

Summing the geometrical series found in the parenthesis, by **360**, (B'), the expression becomes

$$1+\frac{r^{n-1}}{(r+1)^m}-\left(\frac{r}{r+1}\right)^{n-1} \quad . \quad . \quad . \quad . \quad (4).$$

Now since $\frac{r}{r+1}$ is a proper fraction, and therefore less than unity, the value of (4) is positive. Moreover, the negative term, $\left(\frac{r}{r+1}\right)^{n-1}$, must always be less than unity; consequently, no value of r, however great it may be, will render (4) negative. Thus we have shown that if we substitute for x the quantity $\sqrt[n]{P}+1$, or any greater value, the result will be positive in (3); the result will therefore be positive in (2), and also in (1). Hence, by **452**, 2, $\sqrt[n]{P}+1$ is a superior limit of the positive roots in any equation, which was to be proved.

In applying the principle just established, the absolute term must be regarded as the coefficient of x^0; and if the equation is incomplete, the missing terms must be counted, in finding n.

It should be observed also, *that an equation having no negative term can have no positive roots.* For, every positive number substituted for x will render the first member positive. That is, no positive value of x can reduce the first member to zero.

EXAMPLES.

1. Find the superior limit of the positive roots of the equation $x^5+5x^4+2x^3-14x^2-26x+10=0$.

Here $n=3$ and $P=26$. Hence we have, in whole numbers,

$$\sqrt[n]{P}+1=\sqrt[3]{26}+1=4, \textit{ Ans.}$$

2. Find the superior limit of the positive roots of the equation, $x^4 + 5x^3 - 25x^2 - 12x + 68 = 0$. *Ans.* 6.

3. Find the superior limit of the positive roots of the equation, $x^4 - 5x^2 - 9x + 12 = 0$. *Ans.* 4.

4. Find the superior limit of the positive roots of the equation, $x^3 + x^2 + 3x - 8 = 0$. *Ans.* 3.

454. To determine the superior limit of the negative roots of an equation, numerically considered,

Change the signs of the alternate terms, counting the missing terms when the equation is incomplete; then apply the preceding rule.

For, according to **444**, the positive roots in the new equation will be numerically the negative roots in the given equation.

EXAMPLES.

1. Find the superior limit of the negative roots of the equation, $x^3-3x^2+5x+7=0$. *Ans.* $\sqrt[3]{7}+1=3$, in whole numbers.

2. Find the superior limit of the negative roots of the equation, $x^4 - 15x^2 - 10x + 24 = 0$. *Ans.* 5.

3. Find the superior limit of the negative roots of the equation, $x^6 - 3x^5 + 2x^4 + 27x^3 - 4x^2 - 1 = 0$. *Ans.* 4.

LIMITING EQUATION.

455. If there be one equation whose roots, taken in the order of their values, are intermediate between the roots of another, the former is said to be the *limiting equation* of the latter.

456. Any equation being given, its limiting equation may be formed by putting its first derived polynomial equal to zero.

If $a, b, c, \ldots. k, l$ are the roots of the given equation $X=0$, and $a', b', c', \ldots. k'$ are the roots of the derived polynomial $X_1=0$, each set being arranged in the order of their values, then we are to show that all these roots, taken together, and arranged in the order of their values, will be as follows:

$$a, a', b, b', c, c', \ldots. k, k', l.$$

In both equations, put $x = x' + u$, developing the terms, and arranging the results according to the ascending powers of u.

Observe that X_2 is the first derived polynomial of X_1 ; hence, adopting the same notation as in **436**, we have

$$X = X' + X'_1 u + \frac{X'_2}{2}u^2 + \frac{X'_3}{2\cdot 3}u^3 + \ldots\ldots = 0 \quad . \quad . \quad . \quad (1),$$

$$X_1 = X'_1 + X'_2 u + \frac{X'_3}{2}u^2 + \frac{X'_4}{2\cdot 3}u^3 + \ldots\ldots = 0 \quad . \quad . \quad . \quad (2);$$

where, it will be observed, X', X'_1, X'_2, etc., represent what X, X_1, X_2, etc., become, when x' takes the place of x.

Now suppose $x' = r$; that is, $x = r + u$, r being *any root of the given equation.* Then $X' = 0$; and as X'_1, X'_2, X'_3, now receive *definite values,* the values of X and X_1 may, or may not become zero by giving a particular value to u. Dropping X' from (1), and factoring the result, we have

$$X = u\left(X'_1 + \frac{X'_2}{2}u + \frac{X'_3}{2\cdot 3}u^2 + \ldots\right) \quad . \quad . \quad . \quad (3),$$

$$X_1 = \left(X'_1 + X'_2 u + \frac{X'_3}{2}u^2 + \ldots\right) \quad . \quad . \quad . \quad (4);$$

where the different terms may be essentially positive or negative, according to the values of r and u, upon which they depend.

Now, it is evident that by causing u to diminish numerically, each term after the first, in the parenthesis, may be made as small as we please; and by making u sufficiently small, the *sum* of the terms containing u, in each parenthesis, may be made less than the first term X'_1; in which case the essential sign of the quantity in either parenthesis will depend upon the sign of X'_1. Thus, when u is infinitely small, the signs of the functions, X and X_1, will depend upon the signs of $u\,(X'_1)$ and X'_1, respectively. Hence, when u is negative, X and X_1 will have opposite signs; but when u is positive, X and X_1 will have the same signs.

457. Thus we have shown, *that if we substitute in a given equation* X = 0, *and its first derived polynomial* $\mathrm{X}_1 = 0$, *a quantity* r — u, *which is insensibly less than the root* r, *the results will have opposite signs; but if we substitute the quantity* r + u, *which is insensibly greater than the root* r, *the results will have the same sign.*

458. Consider the quantity substituted in the two functions to be insensibly less than a, the least root of $X = 0$, and let it

increase till it is insensibly greater than a. In passing the root a, the function X will change sign (**452**, 3); hence the signs of the functions will be as follows:

$$\text{For}\quad \begin{cases} x = a - u \\ x = a \\ x = a + u \end{cases} \quad \begin{array}{cc} X & X_1 \\ + & -, \\ 0 & -, \\ - & -, \end{array} \quad \text{or else} \quad \begin{array}{cc} X & X_1 \\ - & +, \\ 0 & +, \\ + & +. \end{array}$$

Now let the substituted quantity increase from $x = a + u$ to $x = b - u$, a value insensibly near to b, the next root of $X = 0$. According to the principle already established (**457**), X and X_1 must now have *opposite signs*. And since X cannot have changed its sign during the change of x from $a + u$ to $b - u$, there must have been a change of sign in the function X_1. Hence, by **452**, 3, one root of $X_1 = 0$ is found between $a + u$ and $b - u$, or between a and b. In like manner it can be shown that $X_1 = 0$ has one root between b and c, one between c and d, and so on. Hence the proposition is proved.

STURM'S THEOREM.

459. The object of Sturm's Theorem is to determine the number of the real roots of an equation, and likewise the places of these roots, or their initial figures when the roots are irrational.

NOTE.—This difficult problem, which for a long time baffled the skill of mathematicians, was first solved by M. Sturm, his solution being submitted to the French Academy in 1829.

460. We have seen (**435**), that the equal roots of an equation may always be found and suppressed. Now let

$$X = x^m + Ax^{m-1} + Bx^{m-2} + \dots + Tx + U = 0$$

represent any equation having no equal roots, and $X_1 = 0$ its first derived polynomial, or its limiting equation.

We will now apply to the functions X and X_1, a process similar to that required for finding their greatest common divisor (**105**), but with this modification, namely: *that we change the signs of the successive remainders, and neither introduce nor reject a negative factor, in preparing for division.*

Donote the successive remainders, *with their signs changed*, by

$R, R_1, R_2, \ldots . R_{n-1}, R_n$. Since the given equation has no equal roots, there can be no common divisor between X and X_1 (**435**); hence, if the process of division be continued sufficiently far, the last remainder, R_n, must be *different from zero, and independent of* x.

Now in the several functions, $X, X_1, R, R_1, R_2 \ldots . R_{n-1}, R_n$, let us substitute for x any number, as h, and having arranged the signs of the results in a row, note the number of variations of signs. Next substitute for x a number, h', greater than h, and again note the number of variations of signs. *The difference in the number of variations of signs, resulting from the two substitutions, will be equal to the number of real roots comprised between* h *and* h'.

This is Sturm's Theorem, which we will now demonstrate.

Let $Q, Q_1, Q_2, \ldots . Q_{n-1}, Q_n$ denote the quotients in the successive divisions. Now in every case, the dividend will be equal to the product of the divisor and quotient, *plus* the true remainder, or *minus* the remainder with its sign changed. Hence,

$$\left.\begin{array}{ll}(1) & X = X_1 Q - R \\ (2) & X_1 = R\,Q_1 - R_1 \\ (3) & R = R_1 Q_2 - R_2 \\ (4) & R_1 = R_2 Q_3 - R_3 \\ & \ldots\ldots \\ (n) & R_{n-2} = R_{n-1} Q_n - R_n\end{array}\right\} \quad \ldots\ldots \quad (A).$$

From these equations, it follows,

1. *If any number be substituted for* x *in the functions* X, X_1, R, R_1, ... R_n, *no two of them can become zero at the same time.*

For, if possible, let such a value of h be substituted for x as will render X_1 and R zero at the same time. Then the second equation of (A) will give $R_1 = 0$; whence, the third equation will become $R_2 = 0$; and tracing the series through, we shall have, finally, $R_n = 0$, which is impossible.

2. *If any one of the functions becomes zero by substituting a particular value for* x, *the adjacent functions will have contrary signs for the same value.*

For, suppose R_1 in the third equation to become zero; then this equation will reduce to $R = -R_2$. That is, R and R_2 have contrary signs.

Having established these principles, suppose the quantity h, which is to be substituted simultaneously in all the functions, to be a variable, changing by insensible degrees from a less to a greater value. As it passes any of the roots, the function to which this root belongs will reduce to zero, and change sign (**452**, 3).

Let p be a little less than a certain root of $R_2 = 0$ and q a little greater than the same root, the two values being so taken, however, that *no root of* $R_1 = 0$ or $R_3 = 0$ *shall be comprised between them.* As h changes from p to q, R_2 will reduce to zero, and change sign. But neither R_1 nor R_3 will change sign; and since, according to the second principle, these functions have opposite signs when $R_2=0$, they must have opposite signs also when $h=p$ or $h = q$. Now when $h = p$, the arrangement of signs must be

$$\begin{array}{ccc} R_1 & R_2 & R_3 \\ + & \pm & -, \end{array} \quad \text{or} \quad \begin{array}{ccc} R_1 & R_2 & R_3 \\ - & \pm & +; \end{array}$$

giving one variation and one permanence, whichever way the double sign be taken. When $h = q$ the signs must become

$$\begin{array}{ccc} R_1 & R_2 & R_3 \\ + & \mp & -, \end{array} \quad \text{or} \quad \begin{array}{ccc} R_1 & R_2 & R_3 \\ - & \mp & +; \end{array}$$

giving, as before, one variation and one permanence, so that the whole number of variations is neither increased nor diminished.

This reason obviously applies to any function *which is situated between two other functions.* Hence,

3. *When* h *passes a root of any function intermediate between* X *and* R_n, *the number of variations of signs will not be altered.*

As the last function R_n, is independent of x, its sign will not be changed by any substitution for x. It follows, therefore, *that if any change is produced in the number of variations of signs, it must result from a change of the sign of the original function* X.

Let $a, b, c, d, \ldots. l$ be the roots of X, taken in the order of their values. Then the roots of X_1 will be found, the first between a and b, the second between b and c, and so on (**458**). The degree of X_1 is less by 1 than the degree of X; hence, if the degree of X is odd the degree of X_1 will be even, and if the degree of X is even the degree of X_1 will be odd. Now take h less than a; according to **452**, 1, the signs of X and X_1 will be unlike, giving a variation. Let h increase till it is insensibly greater than a; X will change sign, and the variation between X and X_1 will be lost.

Now let h increase till it is insensibly less than b. It will pass the first root of X_1, causing the signs of X and X_1 to be again unlike; but by 3, this change in the sign of X_1 will not alter the whole number of variations in the signs of the functions. Again let h increase till it is insensibly greater than b; X will again change sign, and another variation will be lost. In like manner it may be shown *that the number of variations will be diminished by* 1 *every time* h *passes a root of* X; *hence the truth of the theorem.*

461. If we substitute for x in the several functions $h = -\infty$ and $h' = +\infty$, successively, we shall determine at once the whole number of real roots in the given equation. To ascertain the *signs* of the functions resulting from these substitutions, we require the following principle:

If in any polynomial involving the descending powers of x, *infinity be substituted for* x, *the sign of the whole expression will depend upon the sign of the first term.*

Let $Ax^m + Bx^{m-1} + Cx^{m-2} + Dx^{m-3} + Ex^{m-4} + \ldots$ be the given polynomial. If $x = \infty$, then

$$A > \frac{B}{x} + \frac{C}{x^2} + \frac{D}{x^3} + \frac{E}{x^4} + \ldots \qquad (1),$$

because every term in the second member is less than any assignable quantity, or zero (**188**, 2). Multiplying both members of (1) by x^m, we have

$$Ax^m > Bx^{m-1} + Cx^{m-2} + Dx^{m-3} + Ex^{m-4} + \ldots \qquad (2).$$

That is, when $x = \infty$, the first term of the given polynomial is numerically greater than the sum of all the other terms. Hence the sign of the whole will be the same as the sign of the first term.

462. In the application of Sturm's Theorem, we may always suppress any numerical factor in any of the functions X_1, R, R_1, etc.; for this will not affect the *sign* of the result.

1. Given the equation $x^3 - 3x^2 - 12x + 24 = 0$, to find the number and situation of the real roots.

Suppressing monomial factors, we have for the several functions,

$$\begin{aligned} X &= x^3 - 3x^2 - 12x + 24, \\ X_1 &= x^2 - 2x - 4, \\ R &= x - 2, \\ R_1 &= 4. \end{aligned}$$

Substituting in these functions $x = -\infty$ and $x = +\infty$ successively, we obtain the following results, in respect to signs:

	X	X_1	R	R_1	
For $x = -\infty$,	$-$	$+$	$-$	$+$,	3 variations.
For $x = +\infty$,	$+$	$+$	$+$	$+$,	0 "

Hence, the given equation has 3 real roots.

Since the signs in the given equation present two variations and one permanence, two of the roots must be positive, and the other negative (**446**). To ascertain the situation of the positive roots, let us substitute in the functions, $x = 0$, $x = 1$, $x = 2$, etc., successively, noting the variations of signs in the results.

For $x = 0$,	signs,	$+$	$-$	$-$	$+$,	2 variations.
$x = 1$,	"	$+$	$-$	$-$	$+$,	2 "
$x = 2$,	"	$-$	$-$	$\pm$	$+$,	1 variation.
$x = 3$,	"	$-$	$-$	$+$	$+$,	1 "
$x = 4$,	"	$-$	$+$	$+$	$+$,	1 "
$x = 5$,	"	$+$	$+$	$+$	$+$,	0 "

Since one variation is lost in passing from $x = 1$ to $x = 2$, and one also in passing from $x = 4$ to $x = 5$, one positive root must be situated between 1 and 2, and the other between 4 and 5.

To ascertain the situation of the negative root, substitute $x = 0$, $x = -1$, $x = -2$, etc.; the signs are as follows:

For $x = 0$,	signs,	$+$	$-$	$-$	$+$,	2 variations.
$x = -1$,	"	$+$	$-$	$-$	$+$,	2 "
$x = -2$,	"	$+$	$+$	$-$	$+$,	2 "
$x = -3$,	"	$+$	$+$	$-$	$+$,	2 "
$x = -4$,	"	$-$	$+$	$-$	$+$,	3 "

Hence, the negative root is situated between -3 and -4. The *initial figures* of the several roots will be 1, 4, and -3.

2. Given the equation, $x^4 - 2x^3 - 7x^2 + 10x + 10 = 0$, to find the number and situation of its real roots.

In this example, we have

$$X = x^4 - 2x^3 - 7x^2 + 10x + 10, \quad X_1 = 2x^3 - 3x^2 - 7x + 5,$$
$$R = 17x^2 - 23x - 45, \quad R_1 = 152x - 305, \quad R_2 = 524535.$$

Let $x = -\infty$;	we have	$+$	$-$	$+$	$-$	$+$,	4 variations.
" $x = +\infty$;	"	$+$	$+$	$+$	$+$	$+$,	0 "

Hence, the roots are all real. And since the signs of the given equation give 2 variations and 2 permanences, two of the roots are positive and two are negative. To find the situation of the roots,

Let $x =$	0; we have	$+$	$+$	$-$	$-$	$+$,	2	variations.
$x =$	1; "	$+$	$-$	$-$	$-$	$+$,	2	"
$x =$	2; "	$+$	$-$	$-$	$-$	$+$,	2	"
$x =$	3; "	$+$	$+$	$+$	$+$	$+$,	0	"
$x =$	-1; "	$-$	$+$	$-$	$-$	$+$,	3	"
$x =$	-2; "	$-$	$-$	$+$	$-$	$+$,	3	"
$x =$	-3; "	$+$	$-$	$+$	$-$	$+$,	4	"

Hence, there are two roots situated between 2 and 3, one between 0 and -1, and one between -2 and -3.

We wish now to find limits which will separate the two roots that lie between 2 and 3. Let us transform the given equation into another *whose roots shall be less by* 2. By **443**, the operation will be as follows :

$$\begin{array}{rrrrr|l}
1 & -2 & -7 & +10 & +10 & \underline{2} \\
 & +2 & 0 & -14 & -8 & \\
\hline
 & 0 & -7 & -4, & +2 & = X \\
 & 2 & +4 & -6 & & \\
\hline
 & 2 & -3, & -10 & = X'_1 & \\
 & 2 & +8 & & & \\
\hline
 & 4, & +5 & = \frac{X'_2}{2} & & \\
 & 2 & & & & \\
\hline
 & 6 & = \frac{X'_3}{2 \cdot 3} & & &
\end{array}$$

The transformed equation is therefore

$$V = y^4 + 6y^3 + 5y^2 - 10y + 2 = 0.$$

Now since the two positive roots of the original equation are found between 2 and 3, the two corresponding roots of the transformed equation must lie between 0 and 1. The situation of these roots may be found from V alone, by trials as follows :

Substitute $y =$	0,	.1,	.2,	.3,	.4,	.5,	.6,	.7,	.8.
The signs of V are	$+$	$+$	$+$	$-$	$-$	$-$	$-$	$-$	$+$.

Hence, one root of V lies between .2 and .3, and one between .7 and .8. Consequently the initial figures of the two positive roots in the original equation, are 2.2 and 2.7.

Note.—If we had found the initial figures of the two positive roots of V to be the same, we should have proceeded to transform V, and make similar trials with the result.

We are now prepared to find the roots of an equation to any degree of accuracy, by

HORNER'S METHOD OF APPROXIMATION.

463. In the year 1819, W. G. Horner, Esq., an English mathematician, published a most elegant and concise method of approximating to the roots of a numerical equation of any degree. The process consists in a series of transformations, the roots of each successive equation being less than the roots of the preceding equation by the initial figures of the preceding roots. But in making the several transformations, the initial figures are obtained by *trial division*, as in square and cube root, and not by substitutions, as in the last article.

464. Let us take an equation in the general form, thus:

$$X = x^m + Ax^{m-1} + Bx^{m-2} + \ldots + Tx + U = 0 \quad . \quad . \quad . \quad (1).$$

Let r represent the initial figure or figures of one of the real roots of this equation, as found by Sturm's Theorem, or otherwise. Now let the equation be transformed into another whose roots shall be less by r. Put $x = r + y$; we shall have

$$V = y^m + A'y^{m-1} + B'y^{m-2} + \ldots + T'y + U' = 0 \quad . \quad . \quad (2).$$

In this equation y is supposed to represent a decimal, since r includes at least the *entire* part of the required root. Hence, the terms containing the higher powers of y are comparatively small; neglecting these, we have, approximately,

$$T'y + U' = 0; \text{ or } y = -\frac{U'}{T'}.$$

Denote the first figure of this quotient by s; put $y = s + z$. Transforming (2) into another whose roots shall be less by s, we have

$$V' = z^m + A''z^{m-1} + B''z^{m-2} + \ldots + T''z + U'' = 0.$$

Whence we have, as before,

$$z = -\frac{U''}{T''} = t + z',$$

where t is another figure of the required root. This process may be continued at pleasure, and we shall have, finally,

$$x = r + s + t + \text{etc.}$$

Hence, to solve a numerical equation of any degree, *we first find by Sturm's Theorem, or otherwise, the number of real roots, and also the first figure or figures of each.* We may then approximate to the value of any root by the following

RULE.—I. *Transform the given equation into another whose roots shall be less by the initial figure or figures of the required root.*

II. *Divide the absolute term of the transformed equation by the penultimate coefficient, as a trial divisor, and write the first figure of the quotient as the next figure of the root sought.*

III. *Transform the last equation into another whose roots shall be less than those of the previous equation by the figure last found; and thus continue till the root be obtained to the required degree of accuracy.*

NOTES.—1. The successive transformations required in obtaining any root may all be made in a single operation; and for the sake of perspicuity, the coefficients obtained in each transformation may be marked or numbered.

2. If a trial figure of the root, obtained by any division, reduces the absolute term X', and the penultimate coefficient X'_1, to the same sign, this figure is not the true one, and must be changed.

3. To obtain the negative roots, it will be most convenient to change the signs of the alternate terms of the given equation, and find the *positive* roots of the result; these, with their signs changed, will be the negative roots required.

4. If the penultimate coefficient, T', should reduce to zero in the operation, the next figure of the root may be obtained *by dividing the absolute term,* U', *by the coefficient which precedes* T', *and extracting the square root of the quotient.* For, if T' vanishes, we have, in the transformed equation,

$$S'y^2 + U' = 0; \quad \text{or} \quad y = \sqrt{-\frac{U'}{S'}}.$$

EXAMPLES.

1. Given $x^3 - 2x^2 - 20x - 40 = 0$, to find the approximate value of x.

By Sturm's Theorem we find that this equation has only one real root, the initial figure being 6. We now obtain the decimal part, to 2 places, as follows.

OPERATION.

-2	-20	-40 | 6.23
6	24	24
$+4$	$+4$	(1) -16
6	60	13.448
10	(1) 64	(2) $-$ 2.552
6	3.24	
(1) 16	67.24	
0.2	3.28	
16.2	(2) 70.52	
.2		
16.4		
.2		
(2) 16.6		

EXPLANATION.—We first transform the given equation into another whose roots are less by 6, using the method of Synthetic Division, explained in **443**. The coefficients of the transformed equation are 16, 64, and -16, marked (1) in the operation. Dividing the absolute term -16, taken with the contrary sign, by the penultimate coefficient 64, we obtain .2, the next figure of the root.

We next transform the equation whose coefficients are marked (1), into another whose roots are less by .2, the resulting coefficients being marked (2). Dividing 2.552 by 70.52, we obtain .03, the next figure of the root. The operation may thus be continued till the root is obtained to any required degree of accuracy.

2. Given $x^4 + x^3 - 30x^2 - 20x - 20 = 0$, to find one value of x.

By Sturm's Theorem, we find the initial figures of the two

real roots to be 5 and − 5. Changing the signs of the alternate terms of the equation, we obtain the decimal part of the negative root, by the following

OPERATION.

IV.	III.	II.	I.	
1 −1	−30	+20	−20	5.731574
5	20	−50	−150	
+4	−10	−30	(1) −170.0000	
5	45	175	159.7071	
9	+35	(1) +145.000	(2) −10.2929	
5	70	83.153	9.7727	
14	(1) 105.00	228.153	(3) −.5202	
5	13.79	93.149	.3304	
(1) 19.0	118.79	(2) 321.302	(4) −.1898	
0.7	14.28	4.455	.1653	
19.7	133.07	325.757	(5) −245	
.7	14.77	4.474	232	
20.4	(2) 147.84	(3) 330.23	(6) −13	
.7	.65	.15	13	
21.1	148.49	330.38	0	
.7	.65	.15		
(2) 21.8	149.14	(4) 330.5		
	.65	.1		
	(3) 150	330.6		
		.1		
	(4) 2	(5) 331		
		(6) 33		

Ans. − 5.731574.

EXPLANATION.—We proceed as in the preceding example till we obtain the terms marked (2), in the operation. Dividing 10.2929 by 321.302, we obtain .03 for the next figure of the root.

At this point we commence to apply decimal contractions, according to the principles employed in the contracted method of cube root (**243**). Let it be observed, *that each contracted term in the operation contains one redundant figure at the right.*

Commencing with column IV, we have $21.8 \times .03 = .65$, which added to column III gives 148.49. Then $148.49 \times .03 = 4.455$, which added to column II gives 325.757. Then $325.757 \times .03 = 9.7727$, which added to column I gives $-.5202$. Again adding .65 to column III, we have 149.14. Then $149.14 \times .03 = 4.474$, which added to column II gives 330.23, after dropping *one place.* Again, adding .65 to column III gives 150 after dropping *two places.* In like manner we continue till the work is finished.

NOTE.—Observe, as a *general rule*, to contract the several columns, for each root figure, as follows: Column I, 0 place; column II, 1 place; column III, 2 places; column IV, 3 places; and so on.

Find the real roots of the following equations:

3. $x^3 + 2x^2 - 23x - 70 = 0.$ *Ans.* 5.1345787253.

4. $x^3 - x^2 + 70x - 300 = 0.$ *Ans.* 3.7387936782.

5. $x^3 + x^2 - 500 = 0.$ *Ans.* 7.6172797559.

6. $x^3 - x^2 - 40x + 108 = 0.$ *Ans.* $\begin{cases} 3.3792053825, \\ 4.5875359541, \\ -6.9667413367. \end{cases}$

7. $x^3 - 4x^2 - 24x + 48 = 0.$ *Ans.* $\begin{cases} 1.7191292611, \\ 6.5461457261, \\ -4.2652749871. \end{cases}$

8. $x^4 + x^3 + x^2 - x - 500 = 0.$ *Ans.* $\begin{cases} 4.4604168201, \\ -4.9296646474. \end{cases}$

9. $x^4 - 9x^3 - 11x^2 - 20x + 4 = 0.$ *Ans.* $\begin{cases} .1796840250, \\ 10.2586086356. \end{cases}$

10. $x^4 - 12x^2 + 12x - 3 = 0.$ *Ans.* $\begin{cases} 2.8580833082, \\ .6060183069, \\ .4432769396, \\ -3.9073785547. \end{cases}$

11. $x^5 - 10x^3 + 6x + 1 = 0.$ *Ans.* $\begin{cases} -3.0653157913, \\ -\ .6915762805, \\ -\ .1756747993, \\ .8795087084, \\ 3.0530581627. \end{cases}$

NOTE.—Full solutions of the examples above may be found in the Key.

EGREE

=.65. r

03 = 4

757 × 5

ain sin

3 = 4

hence

dace. :

7253

782

559.

25,

41,

1,

1,